MEROMORPHIC FUNCTIONS

Recall that a meromorphic function on an open set $U \subseteq \mathbb{C}$ is a quotient of two holomorphic functions

$$f = \frac{g(z)}{h(z)} \qquad g, h \text{ holomorphic on } U$$

<u>Rmk</u>: So, a meromorphic function is not defined to be a function. We will see later that they can actually be interpreted as such.

Now, for an $z_0 \in U$ we can write

$$f = \frac{(z-z_0)^e g_0(z)}{(z-z_0)^f h_0(z)} = (z-z_0)^{e-f} \frac{g_0(z)}{h_0(z)} \qquad \begin{array}{l} g_0(z_0) \neq 0 \\ h_0(z_0) \neq 0 \end{array}$$

In other words, we can write

$$f = (z-z_0)^m f_0(z) \qquad f_0(z) \text{ holomorphic around } z_0 \text{ and } f_0(z_0) \neq 0$$

<u>Def</u>: ORDER of a MEROMORPHIC FUNCTION

With the previous notation, we define the ORDER of f at z_0 as

$$\operatorname{ord}_{z_0}(f) = m.$$

- If $m > 0$: z_0 is called a ZERO of order m for f
- If $m < 0$: z_0 is called a POLE of order $-m$ for f.

<u>Easy but important</u>: $\operatorname{ord}_{z_0}(f \cdot g) = \operatorname{ord}_{z_0}(f) + \operatorname{ord}_{z_0}(g)$.

Observe moreover that if we write $f = (z-z_0)^m f_0(z)$, since f_0 is holomorphic around z_0, we have an analytic expansion

$$f_0(z) = \sum_{n=0}^{\infty} a_{0,n}(z-z_0)^n$$

Hence we can write

$$f = \sum_{n=m}^{\infty} a_n (z-z_0)^n \quad [a_n = a_{0,n-m}]$$

$$= a_m (z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \dots$$

This is a power series with finitely many terms with negative exponents (when $m<0$ i.e. z_0 is a pole), called a LAURENT SERIES.

This can be generalized to Riemann surfaces:

Def: MEROMORPHIC FUNCTION

A meromorphic function on a Riemann surface S is a collection $f_i = \frac{g_i}{h_i}$ of meromorphic functions on each chart $U_i \subseteq S$. On $U_i \cap U_j$ we require that $f_i = f_j$, which is defined as $g_i h_j = g_j h_i$.

All the notions of order, zeroes, poles and Laurent series expansion extend to Riemann surfaces, using local coordinates.

Now we show that meromorphic functions can actually be thought of as functions, not to $\mathbb{C}$ but to $\mathbb{P}^1$.

We consider $\mathbb{P}^1$ as $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, where $\mathbb{C} = \{x_0 \neq 0\} = \{[1, z]\}$ and $\infty = [0,1]$.

- <u>FROM MEROMORPHIC FUNCTIONS to MAPS to $\mathbb{P}^1$.</u>

Then, let X be a Riemann surface and f a meromorphic function on X, given by a collection of meromorphic functions $f = \frac{g_i}{h_i}$ on charts $U_i \subseteq X$. Then we can define a map

$$f : X \to \mathbb{P}^1$$

by gluing together the maps

$$f : U_i \to \mathbb{P}^1 \qquad f(z) = \left[1, \frac{g_i(z)}{h_i(z)}\right] = \left[\frac{h_i(z)}{g_i(z)}, 1\right] = [h_i(z), g_i(z)]$$

We claim that this is an holomorphic function.

- <u>Well-defined</u>: the maps $f : U_i \to \mathbb{P}^1$ are defined without problems outside the zeroes of g_i and h_i. We should explain what happens at these points. In general, let $z_0 \in U_i$. Then we can write

$$g_i(z) = (z - z_0)\, g_0(z) \qquad h_i(z) = (z - z_0)\, h_0(z)$$

with $g_0(z_0) \neq 0$, $h_0(z_0) \neq 0$. So, in a small punctured disk $\Delta \setminus \{z_0\}$ around z_0 we can write

$$f(z) = \left[1, (z-z_0)^m \left(\frac{g_0(z)}{h_0(z)}\right)\right] \qquad m = e-f$$
$$= \mathrm{ord}_{z_0}(f)$$
$$= \left[(z-z_0)^{-m}\left(\frac{h_0(z)}{g_0(z)}\right), 1\right]$$
$$= \left[h_0(z), (z-z_0)^m g_0(z)\right]$$
$$= \left[(z-z_0)^{-m} h_0(z), g_0(z)\right]$$

Now we have three cases

- if $m=0$: then all expressions extend naturally to Δ, for example as
$$f(z_0) = \left[1, \frac{g_0(z_0)}{h_0(z_0)}\right]$$
- if $m>0$: then the 1st and 3rd expressions extend to Δ as
$$f(z_0) = [1, 0] = \{0\}$$
- if $m<0$: then the 2nd and 4th expressions extend to Δ as
$$f(z_0) = [0, 1] = \{\infty\}$$

In this way, the maps $f: U_i \to \mathbb{P}^1$ are well defined everywhere. Now we should check that they coincide on the intersection $U_i \cap U_j$. But this is easy because
$$\left[1, \frac{g_i(z)}{h_i(z)}\right] = \left[1, \frac{g_j(z)}{h_j(z)}\right]$$
since $g_i(z)h_j(z) = g_j(z)h_i(z)$.

Holomorphic: it is enough to check that each map $f: U_i \to \mathbb{P}^1$ is holomorphic. Let $z_0 \in U_i$ and let $m = \mathrm{ord}_p(f)$. If $m \geq 0$ we can see this map in the affine local coordinate on $\mathbb{P}^1$ as

$$f(z) = (z - z_0)^m \frac{g_0(z)}{h_0(z)}$$

and this is clearly holomorphic. Moreover

$$\mathrm{mult}_{z_0}(f) = m = \mathrm{ord}_{z_0}(f).$$

If instead $m < 0$ we know that $f(z_0) = \infty$ and we can use the local coordinate around ∞ to write

$$f(z) = (z - z_0)^{-m} \frac{h_0(z)}{g_0(z)}$$

and this is again holomorphic. Moreover

$$\mathrm{mult}_{z_0}(f) = -m = -\mathrm{ord}_{z_0}(f).$$

- FROM MAPS to $\mathbb{P}^1$ to MEROMORPHIC FUNCTIONS

Let X be a Riemann surface and $f: X \longrightarrow \mathbb{P}^1$ a holomorphic map to $\mathbb{P}^1$.
Let $p \in X$, $\Delta \subseteq X$ a small neighborhood of p, that we identify with a small open disk around 0 in $\mathbb{C}$, and let z be the corresponding local coordinate. We have three cases:

- $f(p) \neq 0, \infty$: then on Δ we can look at f as an holomorphic map $f: \Delta \to \mathbb{C}$ s.t. $f(0) \neq 0$.

This gives the meromorphic function

$$f = \frac{f(z)}{1} \text{ on } \Delta \qquad \text{and } \operatorname{ord}_p(f) = 0.$$

- $f(p) = 0$: then as above we can write

$$f = \frac{f(z)}{1} = \frac{z^m f_0(z)}{1}, \quad f_0(0) \neq 0 \text{ and } \quad \begin{aligned} \operatorname{ord}_p(f) &= \\ &= m \\ &= \operatorname{mult}_p(f). \end{aligned}$$

- $f(p) = \infty$: we can take the standard chart $U_1 = \{x_1 \neq 0\}$ around ∞ and assume that we have an holomorphic map $f: \Delta \to U_1$

We can write this as $f(z) = z^m f_0(z)$, $f_0(0) \neq 0$ and $m = \operatorname{mult}_p(f)$.

Now on the punctured disk $\Delta \setminus \{0\}$, we can also look at this as the function

$$\begin{aligned} f: \Delta \setminus \{0\} &\longrightarrow U_0 \setminus \{0\} = U_1 \setminus \{\infty\} \\ z &\longmapsto \frac{1}{z^m f_0(z)} \end{aligned}$$

so we get the meromorphic function

$$f = \frac{1}{z^m f_0(z)}, \quad f_0(0) \neq 0, \qquad \begin{aligned} \operatorname{ord}_p(f) &= -m \\ &= -\operatorname{mult}_p(f) \end{aligned}$$

All these meromorphic functions glue together to a meromorphic function on the whole of X.

Thus, we have proved the following fundamental

<u>Fact</u>: For any Riemann surface X there is a correspondence

$$\boxed{\left\{\begin{matrix}\text{meromorphic}\\ \text{functions on } X\end{matrix}\right\} \cong \left\{\begin{matrix}\text{holomorphic maps}\\ X \longrightarrow \mathbb{P}^1\end{matrix}\right\}}$$

and under this correspondence we have that

$$\text{ZEROES of } f = f^{-1}(0)$$
$$\text{POLES of } f = f^{-1}(\infty)$$

$$\operatorname{ord}_p(f) = \begin{cases} \operatorname{mult}_p(f) & \text{if } p \text{ is a zero} \\ -\operatorname{mult}_p(f) & \text{if } p \text{ is a pole}\end{cases}$$

<u>Rmk</u>: The above correspondence is <u>slightly incorrect</u>. Indeed, if $\lambda \in \mathbb{C}$, then the constant meromorphic function $\frac{\lambda}{1}$ on X gives rise to the constant map function $\lambda : X \longrightarrow \mathbb{C}$ $\lambda(p) = \lambda$.

However there is no constant that is ∞. Hence the correspondence is between meromorphic functions on X and maps $f : X \to \mathbb{P}^1$ which are not identically $f(p) = \infty$.

<u>Rmk</u>: In particular, holomorphic functions correspond to meromorphic functions without poles.

Prop: Let X be a compact Riemann surface and f a nonconstant meromorphic function. Then

$$\sum_{p\in X} \mathrm{ord}_p(f) = 0 .$$

proof: We can write

$$\sum_{p\in X} \mathrm{ord}_p(f) = \sum_{\text{zeroes}} |\mathrm{ord}_p(f)| - \sum_{\text{poles}} |\mathrm{ord}_p(f)| .$$

If we interpret f as a map $f: X \to \mathbb{P}^1$, we know that

$$\sum_{\text{zeroes}} |\mathrm{ord}_p(f)| = \sum_{p\in f^{-1}(0)} \mathrm{mult}_p(f) = \deg(f)$$

$$\sum_{\text{poles}} |\mathrm{ord}_p(f)| = \sum_{p\in f^{-1}(\infty)} \mathrm{mult}_p(f) = \deg(f). \quad \square$$

Lemma: Let X be a compact Riemann surface. Then any nonconstant rational function has at least a pole.

proof: otherwise, gives a map $f: X \to \mathbb{P}^1$ s.t. $f^{-1}(\infty) = \emptyset$. However any map between compact Riemann surface is either constant or surjective. $\square$

Lemma: Let X be a compact Riemann surface which admits a rational function with exactly one pole (counted with multiplicity). Then $X \cong \mathbb{P}^1$!

proof: We can interpret f as a map $f: X \to \mathbb{P}^1$ of degree $|f^{-1}(\infty)| = 1$. Hence f is bijective and an isomorphism. $\square$

Example: RATIONAL FUNCTIONS on $\mathbb{P}^1$

We can classify all rational functions on $\mathbb{P}^1$. The important point is that on $\mathbb{P}^1$ we can find functions with arbitrary zeroes and poles at each point. Indeed if we consider $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ then for any $a \in \mathbb{C}$

$f = \frac{(z-a)}{1}$ has a simple zero at a and a simple pole at ∞.

$f = \frac{1}{(z-a)}$ has a simple pole at a and a simple zero at ∞.

and, as the previous Lemma shows, this can happen only on $\mathbb{P}^1$.

Now, let f be an arbitrary non constant rational function and let

$\{p_1, \dots, p_n\} = \{\text{zeroes of } f \text{ on } \mathbb{C}\}$ $\quad ord_{p_i}(f) = e_i$

$\{q_1, \dots, q_m\} = \{\text{poles of } f \text{ on } \mathbb{C}\}$ $\quad ord_{q_i}(f) = -f_i$

Then the meromorphic function

$$\frac{(z-q_1)^{f_1} \cdots (z-q_m)^{f_m}}{(z-p_1)^{e_1} \cdots (z-p_n)^{e_n}} f = g$$

has no poles or zeroes on $\mathbb{C}$. So, the only zeroes or poles must be at ∞. However, we know that if g is nonconstant, it must have at least a pole and at least a 0. (previous lemmas) Hence ∞ must be both a zero and a pole, absurd! Hence g is constant.

Another possible approach for this is the following: we can look at g as a holomorphic function

$$g : \mathbb{C} \to \mathbb{C} \qquad g(z) = \sum_{n=0}^{\infty} a_n z^n$$

Now we use that g is meromorphic also at ∞: this means that

$$g\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^{-n}$$

hould be a Laurent series, so that it has finitely many negative terms. In other words, g is a polynomial, but since g has no zeroes, it must be a constant.

In any case, we see that any meromorphic function on $\mathbb{P}^1$ is of the form

$$f = \lambda \cdot \frac{(z-p_1)^{e_1} \cdots (z-p_n)^{e_n}}{(z-q_1)^{f_1} \cdots (z-q_m)^{f_m}} = \frac{\text{polynomial}}{\text{polynomial}}$$

In other words, any meromorphic function is actually a <u>RATIONAL FUNCTIONS</u>.

Observe that on $\mathbb{C}$ there are many meromorphic functions that are not rational, for example the exponential function $\exp(z) = \sum \frac{z^n}{n!}$

However, the compactness of $\mathbb{P}^1$ forces the algebraicity.

Example: MEROMORPHIC FUNCTIONS on PLANE CURVES

Let $C = \{F(X,Y,Z)=0\}$ be a smooth plane projective curve. Consider two lines

$$L_1 = \{\ell_1(X,Y,Z)=0\}, \quad L_2 = \{\ell_2(X,Y,Z)=0\}$$

where ℓ_1, ℓ_2 are two linear forms. Assume that $C \neq L_1, L_2$. Then we claim that $\frac{\ell_1(X,Y,Z)}{\ell_2(X,Y,Z)}$ is a meromorphic function on C and moreover that

$$\boxed{\mathrm{ord}_p\left(\frac{\ell_1(X,Y,Z)}{\ell_2(X,Y,Z)}\right) = \mu_p(C,L_1) - \mu_p(C,L_2)}$$

In particular our proposition of before is a consequence of Bezout's Theorem:

$$\sum_{p\in C} \mathrm{ord}_p\left(\frac{\ell_1(X,Y,Z)}{\ell_2(X,Y,Z)}\right) = \sum_{p\in C} \mu_p(C,L_1) - \sum_{p\in C} \mu_p(C,L_2)$$

$$= \deg(C) - \deg(C) = 0$$

Let's prove this statement.

We have the three affine charts

$U_1 = \{X \neq 0\}$, $U_2 = \{Y \neq 0\}$, $U_3 = \{Z \neq 0\}$

with corresponding affine coordinates

$\left(\frac{Y}{X}, \frac{Z}{X}\right)$ on U_1, $\left(\frac{X}{Y}, \frac{Z}{Y}\right)$ on U_2, $\left(\frac{X}{Z}, \frac{Y}{Z}\right)$ on U_3

Then $\frac{e_1(X,Y,Z)}{e_2(X,Y,Z)}$ corresponds to

$$\frac{e_1(X,Y,Z)}{e_2(X,Y,Z)} = \frac{X \cdot e_1(1, \frac{Y}{X}, \frac{Z}{X})}{X \cdot e_2(1, \frac{Y}{X}, \frac{Z}{X})} = \frac{e_1(1, \frac{Y}{X}, \frac{Z}{X})}{e_2(1, \frac{Y}{X}, \frac{Z}{X})} \text{ on } U_1$$

$$= \frac{Y \cdot e_1(\frac{X}{Y}, 1, \frac{Z}{Y})}{Y \cdot e_2(\frac{X}{Y}, 1, \frac{Z}{Y})} = \frac{e_1(\frac{X}{Y}, 1, \frac{Z}{Y})}{e_2(\frac{X}{Y}, 1, \frac{Z}{Y})} \text{ on } U_2$$

$$= \frac{Z \cdot e_1(\frac{X}{Z}, \frac{Y}{Z}, 1)}{Z \cdot e_2(\frac{X}{Z}, \frac{Y}{Z}, 1)} = \frac{e_1(\frac{X}{Z}, \frac{Y}{Z}, 1)}{e_2(\frac{X}{Z}, \frac{Y}{Z}, 1)} \text{ on } U_3$$

Now we need to prove the statement about the order. For simplicity, we restrict to the case $e_1 = X$, $e_2 = Y$. Now restrict to the chart $U_3 = \{Z \neq 0\}$ for example, with affine coordinates $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$.

Then $\frac{X}{Y} = \frac{(X/Z)}{(Y/Z)} = \frac{x}{y}$ and now x, y are holomorphic functions on $\mathbb{C}$.

It is enough to prove

$$\mathrm{ord}_p(x) = \mu_p(x, C)$$

But we know this already because

$$\underbrace{\mathrm{ord}_p(x) = \mathrm{mult}_p((x,y) \mapsto x)}_{\text{order of zero is the same of multiplicity of corresponding map}} \underbrace{=}_{\text{proved in class}} \mu_p(x, C).$$

$\square$