• Maps of compact Riemann surfaces

Now we are going to specialize to maps between COMPACT RIEMANN SURFACES. Those are those that we are interested in, since they correspond to projective algebraic curves.

Let $f: S_1 \to S_2$ be a nonconstant holomorphic map between compact, connected R.S.

(1) f is surjective

proof: since f is holomorphic and nonconstant, it is open so that $f(S_1)$ is open. Since S_1 is compact, $f(S_1)$ is compact and since S_2 is Hausdorff $f(S_1)$ is closed. Hence $f(S_1)$ is open and closed and since S_2 is connected, $f(S_1) = S_2$. □

(2) The fibers of f are finite.

proof: Let $q \in S_2$: $f^{-1}(q)$ is closed, hence compact as S_1 is compact. We are going to prove that it is discrete. Then it will be finite. Let $p \in f^{-1}(q)$: then there is a local coordinate z in an open neighborhood U of p s.t. f looks like $f(z) = z^m$. Then the fiber $f^{-1}(q)$

corresponds to $\{z^m = 0\}$ and the only point in the set is $z=0$. Hence $f^{-1}(q) \cap U = \{p\}$. $\square$

(3) The ramification points of f are finite

proof: the idea is again to prove that the set of ramification points is compact and discrete. Since S_1 is compact, it is enough to prove that it is closed and discrete. Let $p \in S_1$ be a ramification point then we can find again a local coordinate z on an open neighborhood U s.t. $f(z) = z^m$. Then the ramification points inside U are the points s.t.

$$U \cap \left\{\begin{matrix}\text{ramification} \\ \text{points}\end{matrix}\right\} = \{z \in U \mid f(z) = f'(z) = 0\}$$
$$= \{z \in U \mid z^m = m \cdot z^{m-1} = 0\}$$

we see at the same time that this set is closed and that $U \cap \{\text{ram points}\} = \{z = 0\} = \{p\}$, so it is discrete. $\square$

Now we can define the most important invariant of an holomorphic map between compact Riemann surfaces.

Prop/Def Let $f: S_1 \longrightarrow S_2$ be a nonconstant holomorphic map between compact connected Riemann surfaces.

Then for any $q \in S_2$ the number of points in the fiber, counted with multiplicity, is constant:

$$d(q) = \sum_{p \in f^{-1}(q)} \mathrm{mult}_p(f)$$

This is called the DEGREE of the map.

proof: We will show that d is locally constant. Let $q \in S_2$ and let $f^{-1}(q) = \{p_1, \dots, p_n\}$. Then we can find charts $U_i \subseteq S_1$ around each of the p_i and $V \subseteq S_2$ around q such that in the corresponding local coordinates, f looks like $f(z) = z^{e_i}$ around each p_i, $e_i = \mathrm{mult}_p(f)$.
Now, since S_1, S_2 are both compact, by shrinking V if needed, we can assume that $f^{-1}(V) \subseteq U_1 \cup \dots \cup U_n$ (exercise in set topology).
Now we show that the function is constant on V. Let $q' \in V$ then $f^{-1}(q) \subseteq U_1 \cup \dots \cup U_n$. Then we can simply count the points of the fiber in each U_i: on each of those the map looks like $f(z) = z^{e_i}$

and it is now clear that $f^{-1}(a) = \begin{cases} \text{one pt of mult } e_i & , \text{if } a = 0 \\ e_i \text{ distinct pts} & , \text{if } a \neq 0 \end{cases}$

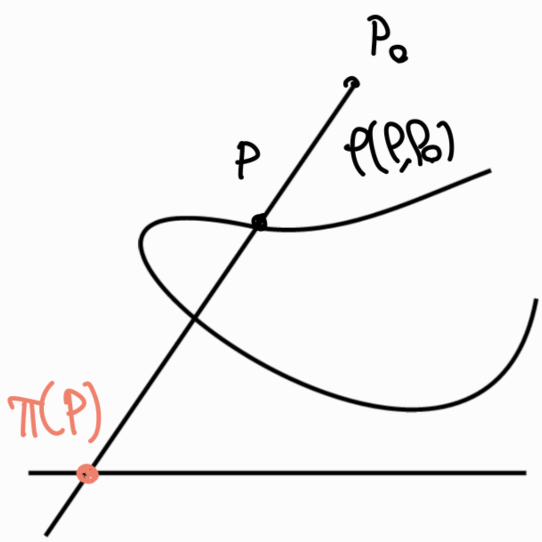
In any case, the sum of pts with multiplicity is e_i. Hence $d(q') = \sum e_i \ \forall q' \in V$. □

Example: SMOOTH PROJECTIVE PLANE CURVES

Let $C = \{F(X,Y,Z)=0\} \subseteq \mathbb{P}^2(\mathbb{C})$ be a smooth plane projective curve of degree d. It can be proven that C is connected in the analytic topology so that it is a connected, compact Riemann surface.

Consider a point $P_0 \in \mathbb{P}^2(\mathbb{C}) \setminus C$, a line L not passing through P_0 and take the linear projection

$$\pi_{P_0} : C \longrightarrow L$$
$$P \longmapsto \ell(P,P_0) \cap L$$



This is a holomorphic map of Riemann surfaces: up to a change of coordinates we can assume that $P_0 = [0,1,0]$ and $L = \{Z=0\}$. Then the projection is

$$\pi_{P_0}([X,Y,Z]) = [X,Z]$$

Since $P_0 \notin C$, it must be that $C \subseteq \{X \neq 0\} \cup \{Z \neq 0\}$ in the chart $\{Z \neq 0\}$ the map is given in affine coordinates by $\left(\frac{X}{Z}, \frac{Y}{Z}\right) \longmapsto \frac{X}{Z}$, $(x,y) \mapsto x$

which is holomorphic. Same in the other chart.

Let's look now at the multiplicity of this projection at a point $p \in C$. We claim that

$$\text{mult}_p(\pi_{P_0}) = \mu_p(C, \ell(P, P_0))$$

Consider everything in the affine situation:

$C = \{F(x,y) = 0\}$ smooth affine curve

$\pi : C \to \mathbb{C}$, $\pi(x,y) = x$.

Up to translation we can assume that $P = (0,0)$ and then $\ell(P, P_0)$ is the axis $\ell = \{x = 0\}$

Now we need to prove that

$$\text{mult}_{(0,0)}(\pi) = \mu_{(0,0)}(F(x,y), x)$$

We consider two cases:

- $\frac{\partial F}{\partial y}(0,0) \neq 0$: then x is a local coordinate on C around $O = (0,0)$. In terms of this local coordinate, π is simply given by $\pi(x) = x$. Hence $\text{mult}_O(\pi) = 1$. On the other hand we know that the tangent line $T_O C$ is <u>not</u> the axis $\ell = \{x = 0\}$ and since C, ℓ are both smooth at O with distinct tangents, we get $\mu_O(C, \ell) = 1$.

• $\frac{\partial F}{\partial y}(0,0)=0$, $\frac{\partial F}{\partial x}(0,0)\neq 0$: then, up to multiplication by a scalar, we can write

$$F(x,y) = x + G(x,y) \rightarrow \text{terms of order} \geq 2$$

$$= H(y) + x\cdot K(x,y) \qquad H(0)=0 \quad K(0,0)=1$$

note that $H(y)\neq 0$ otherwise the curve would not be smooth, or, if smooth would be the line $\{x=0\}$.

Hence we see that

$$\mu_0(x, F(x,y)) = \mu_0(x, H(y))$$

$$= \text{multiplicity of } y=0 \text{ as a root of } H(y)$$

On the other hand, we know that y is a local coordinate around 0 and by definition of C we have

$$H(y) + x\cdot K(x,y) = 0$$

hence $x = -H(y)\cdot\frac{1}{K(x,y)}$

and since $K(0,0)=1$ we can write it as

$$K(x,y) = 1 + a_1 y + a_2 y^2 + \dots$$

so we write $x = -H(y)\cdot(1 + a_1 y + \dots)$

and we see that the multiplicity of x is the same of the multiplicity of $y=0$ as a root of $H(y)=0$.

With this we can compute the degree of π_{P_0}: take a point $Q \in L$, then

$$\pi_{P_0}^{-1}(Q) = \ell(P_0, Q) \cap C$$

and each point is counted with multiplicity. Then Bezout's Theorem gives that

$$\boxed{\deg \pi_{P_0} = d.}$$