HOLOMORPHIC MAPS of RIEMANN SURFACES

We start with a review of some properties of holomorphic functions in one variable.

Let $U \subseteq \mathbb{C}$ an open neighborhood of 0 and let $f : U \to \mathbb{C}$ be a holomorphic function. We look at the behavior of f around a point $p \in U$.

(1) INVERSE FUNCTION THEOREM

Suppose $f'(p) \neq 0$. Then there are open neighborhoods $U_1 \subseteq \mathbb{C}$ of p and $U_2 \subseteq \mathbb{C}$ of $f(p)$ such that $f : U_1 \to U_2$ is invertible with inverse $f^{-1} : U_2 \to U_1$ holomorphic. We say that f is locally an isomorphism at p.

(2) ROOTS of HOLOMORPHIC FUNCTIONS

Fix a positive integer $k > 0$. Then there is an open neighborhood $U' \subseteq U$ of p and a holomorphic function $g : U' \to \mathbb{C}$ s.t.

$$f(z) = g(z)^k \quad \text{on } U'.$$

We say that f has locally a k-th root at p.

Proof: suppose first that $w = f(p) \neq 0$. choose one root $u \in \mathbb{C}^*$ s.t. $u^k = w$. Consider now the holomorphic map $p: \mathbb{C}^* \to \mathbb{C}^*$, $p(z) = z^k$. We see that $p'(u) = k \cdot u^{k-1} \neq 0$, hence by the inverse function theorem there is an open neighborhood $V \subseteq \mathbb{C}^*$ of $w = p(u)$ and a holomorphic function $\sigma: V \to \mathbb{C}^*$ s.t. $p(\sigma(z)) = z \ \forall z \in V$, i.e. $\sigma(z)^k = z$.

Now consider $U' = f^{-1}(V)$ and the holomorphic function $g = \sigma \circ f$. This is a k-th root of f.

This solves the case when $f(p) \neq 0$. Suppose $f(p) = 0$. Then we can write around p

$$f(z) = (z-p)^m \cdot g(z) \quad , g(p) \neq 0$$

We can then find an m-th root of $g(z)$ by the previous point: $g(z) = h(z)^m$, and then $f(z) = ((z-p) \cdot h(z))^m$.

(3) LOCAL FORM OF HOLOMORPHIC FUNCTION

Suppose f is defined around 0, that is nonconstant around 0, and that $f(0) = 0$. Then there is a local coordinate z around 0 s.t. f has the form

$$\boxed{f(z) = z^m.}$$

proof: proceeding as in the previous proof, we can write $f(z) = (z \cdot h(z))^m$ with $h(0) \neq 0$.
Then $w(z) = z \cdot h(z)$ is holomorphic and $w'(0) \neq 0$.
Hence w is a local isomorphism at 0, i.e. w is a local coordinate at 0. So we can write

$$f(w) = w^m.$$

(4) NONCONSTANT HOLOMORPHIC FUNCTIONS are OPEN

Let $f: U \to \mathbb{C}$ be holomorphic and nonconstant with U connected. Then f is open.

proof: this is a local question, so we can assume that f has the form $f(z) = z^m$. This is open at $\{z \neq 0\}$ because it is a local isomorphism there thanks to the inverse function theorem. At $z = 0$ we need to check that the image of small balls are open. But

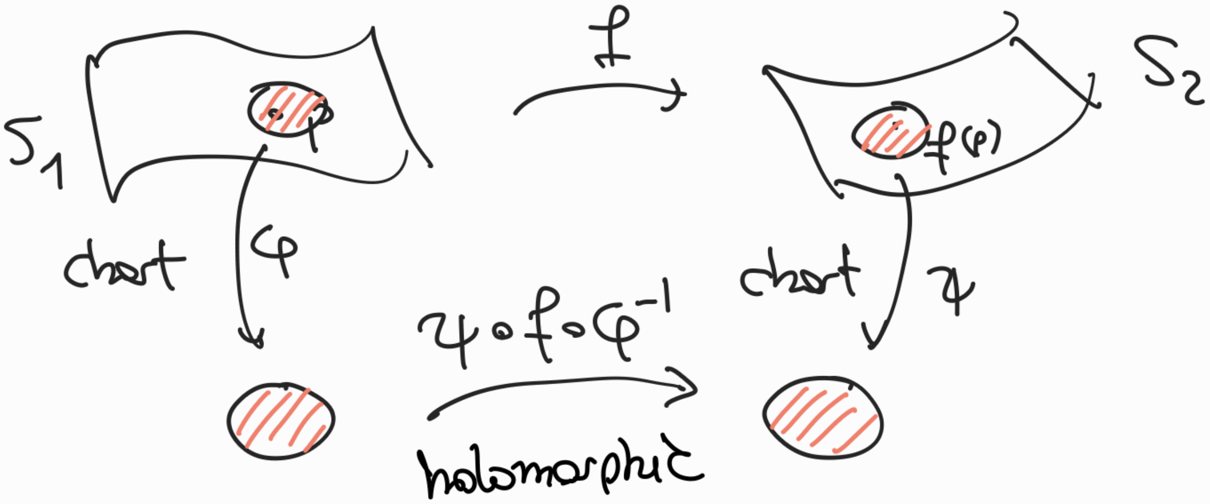
$$f(\{z \mid |z| < \varepsilon\}) = \{z \mid |z| < \varepsilon^m\}$$

which is open. $\square$

Now we can move everything to Riemann surface:

Def: HOLOMORPHIC MAP of RIEMANN SURFACES

A map $f: S_1 \to S_2$ of R.S. is holomorphic if it is holomorphic in local coordinates:



Since all the results about holomorphic functions of before are local, they readily generalizes to holomorphic maps of Riemann Surfaces.

In particular, we know that if $f: S_1 \to S_2$ is a nonconstant holomorphic map between connected Riemann surface, then for each point $p \in S_1$ we can find a local coordinate z on S_1 centered at p $(z(p)=0)$ s.t. f looks like

$$f(z) = z^m$$

Def: MULTIPLICITY of a MAP

In the above notation, m is called the MULTIPLICITY of f at p: $\quad \text{mult}_p(f) := m.$

Rmk: (1) This is independent from the local coordinate z.

(2) If we write f locally as $f = f(z)$, where z is a local coordinate, then

$$\mathrm{mult}_p(f) = m \iff \begin{array}{l} f(p) = f'(p) = \dots = f^{(m-1)}(p) = 0 \\ f^{(m)}(p) \neq 0 \end{array}$$

Def: RAMIFICATION POINTS and BRANCH POINTS

Let $f: S_1 \to S_2$ be a nonconstant map between connected Riemann surfaces as before. A point $p \in S_1$ is a RAMIFICATION POINT if

$$\mathrm{mult}_p(f) > 1$$

A point $q \in S_2$ is a BRANCH POINT if it is the image of a RAMIFICATION POINT.