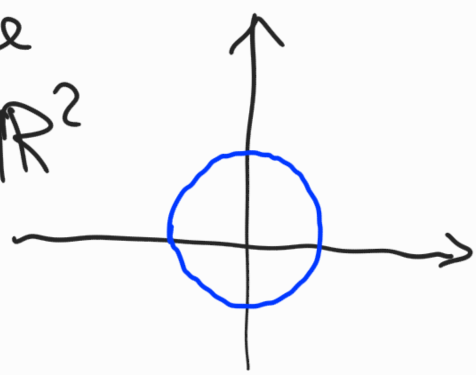
COMPLEX MANIFOLDS and HOLOMORPHIC FUNCTIONS.

The concept of MANIFOLD is one of the fundamental insights of $\geq$ 19th century mathematics.

Before that, geometric objects existed mostly only extrinsically, usually as subsets of the affine space $\mathbb{R}^n$:

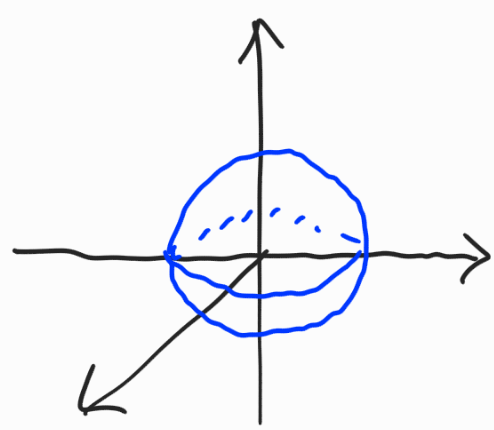
Examples: (1) The circle

$\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\} \subseteq \mathbb{R}^2$



(2) The sphere

$\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$



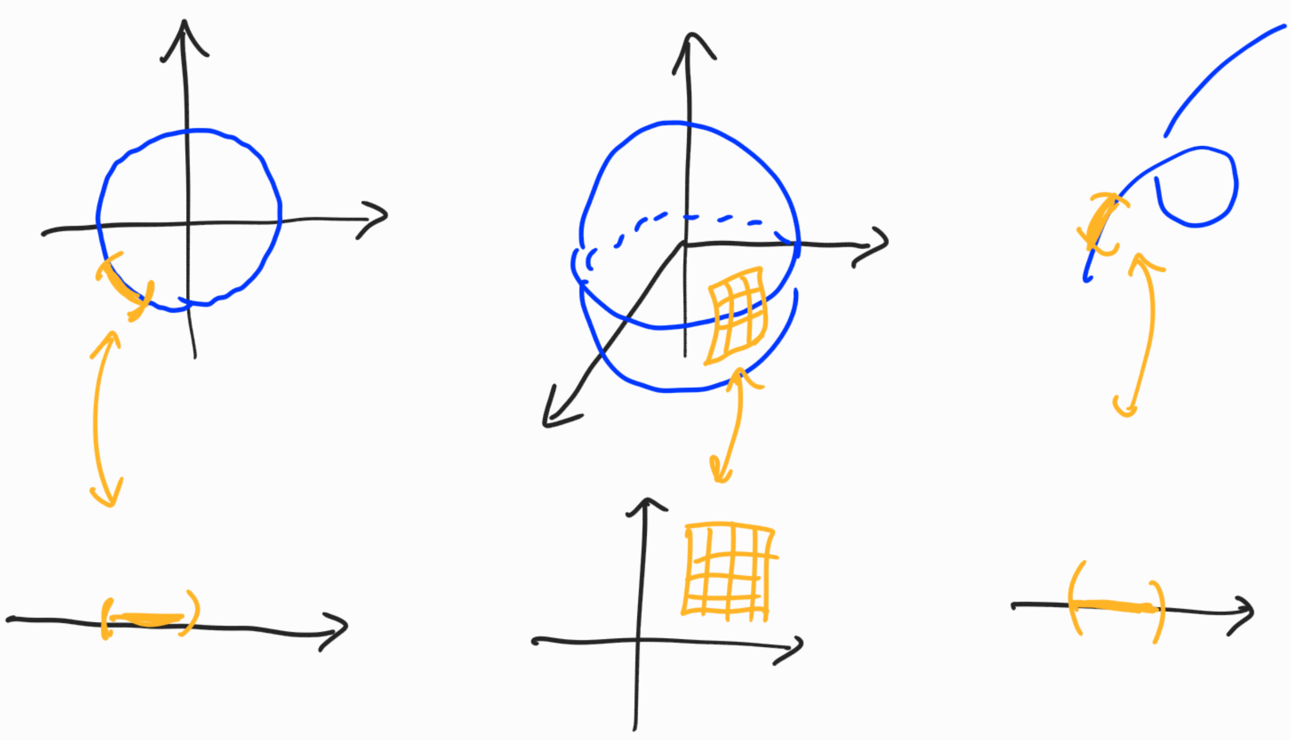
(3) A parametric curve

$\{(t,t^2,t^3) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}$

However, RIEMANN realized that geometric objects can exist INTRINSICALLY, that is, without the need of an AMBIENT SPACE.

This leads to the concept of (SMOOTH) MANIFOLD:

SMOOTH MANIFOLD of dimension n = a topological space that locally looks like the affine space $\mathbb{R}^n$.



Alternatively, this means that on a manifold of dimension n we have local coordinates $(x_1, x_2, \dots, x_n)$ around each point.

In particular we can make CALCULUS (i.e. differentiate and integrate on such manifolds).

Let's now give the formal definition of a manifold.

Def: SMOOTH MANIFOLD

A smooth manifold is a second countable and Hausdorff topological space X together with an open cover

$$X = \bigcup_{i \in I} U_i \quad \text{s.t.}$$

(i) there are homeomorphisms, called CHARTS

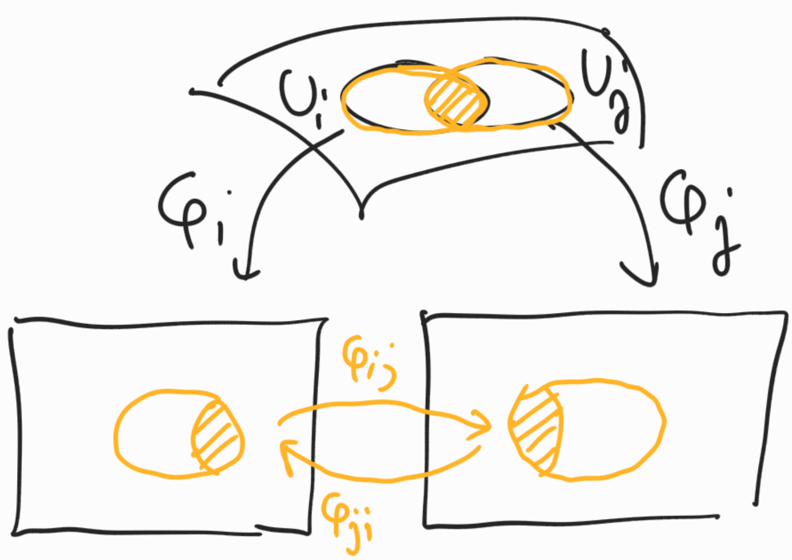
$$\varphi_i : U_i \xrightarrow{\sim} V_i$$

where $V_i \subseteq \mathbb{R}^n$ is an open subset

(ii) those CHARTS are compatible, meaning that the CHANGE of COORDINATES MAPS

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are SMOOTH ($= C^\infty$)



Example: PROJECTIVE SPACE $\mathbb{P}^n$

This is an extremely important example for us. For starters, it is a manifold that does not appear naturally as a subset of something else.

$$\mathbb{P}^n_{\mathbb{R}} = \{ [x_0, \dots, x_n] \}$$ space of $(n+1)$-tuples s.t.

$$\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \neq 0, \quad \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix} \overset{\text{def}}{\iff} \exists \lambda \in \mathbb{R}^* \text{ s.t. } \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$$

Why is this a manifold? Consider the charts

$$U_i = \left\{ \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \middle| \, x_i \neq 0 \right\}.$$ These have homeomorphisms

$$\varphi_i : U_i \xrightarrow{\sim} \mathbb{R}^n \qquad \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} x_0/x_i \\ \vdots \\ \widehat{x_i/x_i} \\ \vdots \\ x_n/x_i \end{bmatrix}$$

from U_i to the affine space $\mathbb{R}^n$ with coordinates $(x_{0i}, x_{1i}, \dots, \hat{x}_{ii}, \dots, x_{ni})$

So, these charts put on each U_i the coordinates

$$x_{ai} = \frac{X_a}{X_i}$$

Now, we should check that the changes of coordinates are smooth. Why should we expect this? Well, on the intersection

$$U_i \cap U_j = \{ x_i \neq 0, x_j \neq 0 \}$$

we have the two sets of coordinates

$$(x_{0i}, x_{1i}, \dots, x_{ni}) \text{ with } x_{ji} \neq 0$$

and $(x_{0j}, x_{1j}, \dots, x_{nj})$ with $x_{ij} \neq 0$

How do we change from one to the other? We see that

$$x_{aj} = \frac{X_a}{X_j} = \frac{X_a}{X_i} \cdot \frac{X_i}{X_j} = \frac{\left(\frac{X_a}{X_i}\right)}{\left(\frac{X_j}{X_i}\right)} = \frac{x_{ai}}{x_{ji}}$$

Thus, to go from one set of coordinates to the other, we simply apply multiplication by $1/x_{ji}$ which is clearly a smooth map.

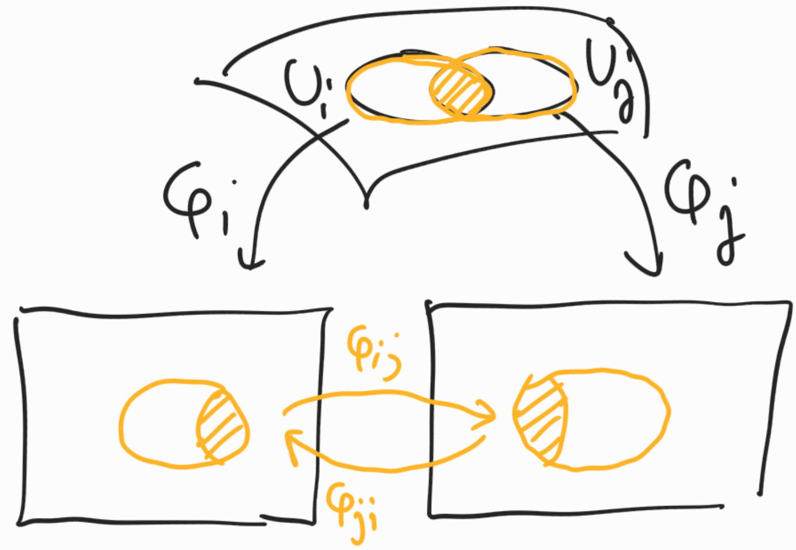
- - - - - - - -

In our course we will not be concerned with smooth manifolds but, rather, with COMPLEX MANIFOLDS. These are spaces that locally look like the complex affine space $\mathbb{C}^n$. Then, we will have to deal with functions that are differentiables in the complex sense: HOLOMORPHIC FUNCTIONS.

The definition is the same as for smooth manifolds, we just change the kind of change of coordinates we are working with:

<u>Def</u>: COMPLEX MANIFOLD
A smooth manifold is a second countable and Hausdorff topological space X together with an open cover
$$X = \bigcup_{i\in I} U_i \quad \text{s.t.}$$
(i) there are homeomorphisms, called CHARTS
$$\varphi_i : U_i \longrightarrow V_i$$
where $V_i \subseteq \mathbb{C}^n$ is an open subset

(ii) these CHARTS are compatible, meaning that the CHANGE of COORDINATES MAPS
$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$
are HOLOMORPHIC



n is called the DIMENSION of the COMPLEX MANIFOLD

Example: (1) $\mathbb{P}^n(\mathbb{C})$ is a complex manifold of dim n

Indeed, we have the charts

$$\varphi_i : U_i = \{x_i \neq 0\} \longrightarrow \mathbb{C}^n$$

$$\begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} x_0/x_i \\ \vdots \\ \widehat{x_i/x_i} \\ \vdots \\ x_n/x_i \end{bmatrix}$$

and the changes of coordinates are given by multiplication by $1/x_j$, which is holomorphic.

(2) Even easier $\mathbb{C}^n$ is a complex manifold of dimension n.

(3) If X is a complex manifold, every open subset $U \subseteq X$ is naturally a complex manifold as well.

Def: RIEMANN SURFACE

A Riemann surface is a (connected) complex manifold of dimension 1.

Examples: (1) $\mathbb{P}^1(\mathbb{C})$ or the Riemann sphere.

This is the most basic Riemann surface. In general the calculations of before, show that $\mathbb{P}^n(\mathbb{C})$ is an n-dimensional complex manifold for any n. Hence, $\mathbb{P}^1(\mathbb{C})$ is a Riemann surface.

In particular, if $\mathbb{P}^1(\mathbb{C}) = \left\{ \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \right\}$ we have two charts

$U_0 = \{x_0 \neq 0\}$ with local coordinate $z = \frac{x_1}{x_0}$

$U_1 = \{x_1 \neq 0\}$ with local coordinate $w = \frac{x_0}{x_1} = \frac{1}{z}$

Since $\mathbb{P}^1(\mathbb{C}) = U_0 \cup \{[0,1]\} \cong \mathbb{A}^1_z \cup \{\infty\}$

the point $[0,1]$ is called point at infinity (w.r.t. the coordinate z).

(2) AFFINE PLANE CURVES

Let $f(z_1, z_2)$ be a nontrivial polynomial in two variables. We consider the affine curve

$$C = \{ f(z_1, z_2) = 0 \} \subseteq \mathbb{C}^2$$

A point $p \in C$ is regular if

$$\left(\frac{\partial f}{\partial z_1}(p), \frac{\partial f}{\partial z_2}(p) \right) \neq 0$$

Then around each regular point the affine curve C

is locally a Riemann surface. More precisely if

$\frac{\partial f}{\partial z_1}(p) \neq 0$, then z_2 is a local coordinate around p

$\frac{\partial f}{\partial z_2}(p) \neq 0$, then z_1 is a local coordinate around p

This works as in the case of smooth manifolds, via the implicit function theorem.

(3) PROJECTIVE PLANE CURVES

Let $F(x_0, x_1, x_2) \in \mathbb{C}[x_0, x_1, x_2]_d$ be an homogeneous polynomial of degree d, such that

$$\left\{ F = \frac{\partial F}{\partial x_0} = \frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} \right\} \subseteq \mathbb{P}^2 \text{ is empty}$$

Then the zero locus $C = \{F = 0\} \subseteq \mathbb{P}^2$ is a Riemann surface. This can be c cked easily on the standard affine charts of $\mathbb{P}^2$.

Rmk: It turns out that the curve $C = \{F = 0\}$ is also connected, but we will prove this later.

(4) HYPERELLIPTIC CURVE

Let's consider a polynomial of even degree $2g+2 \geq 2$

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_{2g+2}) \quad a_i \text{ distinct}$$

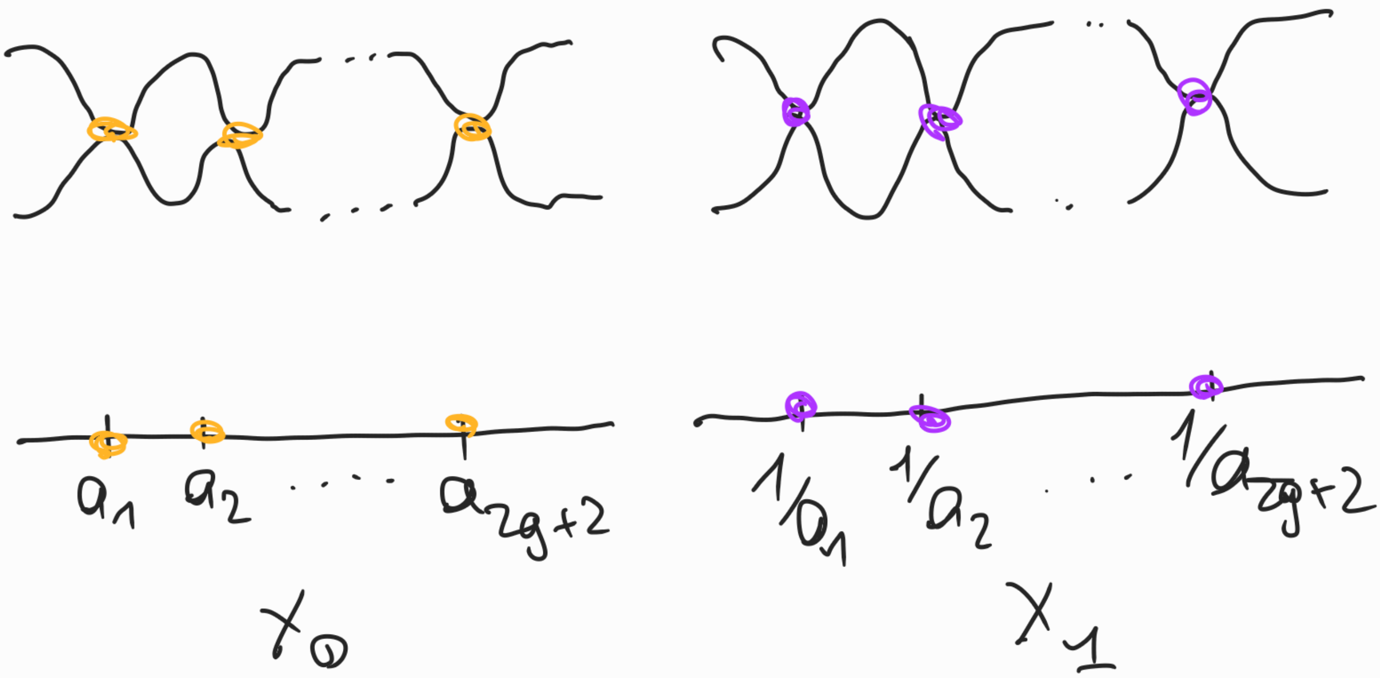
The hyperelliptic curve associated to this polynomial is obtained by gluing two open sets as follows:
First consider the polynomial

$$g(u) = u^{2g+2} \cdot f\left(\frac{1}{u}\right) = (1 - a_1 \cdot u) \cdot \ldots \cdot (1 - a_{2g+2} \cdot u)$$

then we have two affine curves with open subsets

$$X_0 = \{ y^2 = f(x) \} \supseteq U_0 = \{ y^2 = f(x), x \neq 0 \}$$

$$X_1 = \{ v^2 = g(u) \} \supseteq U_1 = \{ v^2 = g(u), u \neq 0 \}$$



Then we can glue X_0, X_1 along U_0, U_1 via the maps

$$\varphi : U_0 \longrightarrow U_1 \qquad \psi : U_1 \longrightarrow U_0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1/x \\ y/x^{g+1} \end{pmatrix} \qquad \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} 1/u \\ v/u^{g+1} \end{pmatrix}$$

Indeed, observe that if $y^2 = f(x)$ and $x \neq 0$, then

$$y^2 = f(x) \Longleftrightarrow y^2 = x^{2g+2} g(1/x) \Longleftrightarrow \left(\frac{y}{x^{g+1}}\right)^2 = g(1/x)$$

The resulting space $X = X_0 \cup X_1$ is a Riemann surface.

Moreover, the two natural maps

$$U_0 \to \mathbb{A}^1_x \quad (x,y) \mapsto x$$
$$U_1 \to \mathbb{A}^1_u \quad (u,v) \mapsto u$$

glue together to a global map

$$f : X \to \mathbb{P}^1.$$

(5) Let $U \subseteq \mathbb{C}^2$ be an open subset and let $f(x,y)$ be an holomorphic function on U and consider the zero locus $C = \{f(x,y) = 0\}$. Suppose that $\left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p)\right) \neq 0$ for all $p \in C$. Then C is a Riemann surface. More precisely if $\frac{\partial f}{\partial x}(p) \neq 0$ then y is a local coordinate around p and viceversa. This can be generalized to a subset $C \subseteq X$ of a 2-dim'l complex manifold X which is in each chart of X cut out by an equation like $f(x,y) = 0$.