2.7. MULTIPLICITY of a CURVE at a POINT

We have seen last time that for two plane curves C, D

C, D intersect transversally at P iff $\mu_P(C,D)=1$ iff C, D are smooth at P with distinct tangents

In particular, we see that when one of the two curves is singular at P, the intersection cannot be transversal. We can make this more precise:

Let $C = \{F(x,y)=0\}$ be an affine curve and $P=(0,0)$. Write F as a sum of its homogeneous terms as

$$F(x,y) = F_0 + F_1 + F_2 + \dots$$

Def: MULTIPLICITY of a CURVE at a POINT

In the above situation, we define

$$\mathrm{mult}_P C = \min\{ m \geq 0 \mid F_m(x,y) \neq 0 \}$$

$$= \min\left\{ m \geq 0 \;\middle|\; \begin{array}{l} \text{one } m\text{-th derivative} \\ \text{of } F \text{ does not vanish} \\ \text{at } P \end{array} \right\}$$

If $\mathrm{mult}_P C = m$, we define the TANGENT CONE to C at P to be the curve $\{F_m(x,y)=0\}$. This is the sum of m lines (with multiplicity) which are called TANGENT LINES to C at P.

This definition was valid for $P=(0,0)$. For any P, just translate or use the second equality. For a projective curve, use an affine chart.

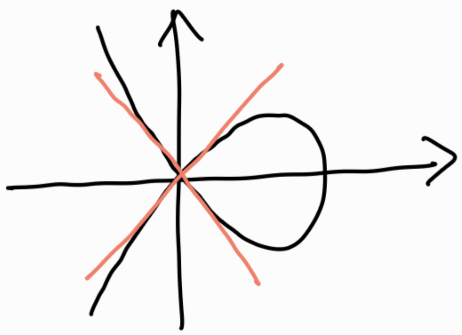
Rmk: · $\mathrm{mult}_P C = 0$ iff $P \notin C$

· $\mathrm{mult}_P C = 1$ iff C is smooth at P and the tangent line is the usual tangent line.

Example: (1) $C = \{ y^2 - x^2 + x^3 = 0 \}$

$\mathrm{mult}_0 C = 2$, tangent cone at 0 $= \{ y^2 - x^2 \}$

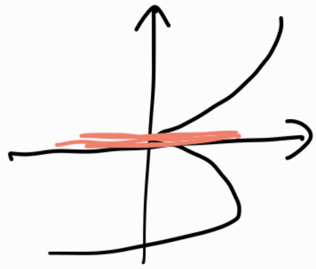
tangent lines $= \{ y - x = 0 \}, \{ y + x = 0 \}$



(2) $C = \{ y^2 - x^3 + y^5 = 0 \}$

$\mathrm{mult}_0 C = 2$, tangent cone at 0 $= \{ y^2 = 0 \}$

tangent lines $= \{ y = 0 \}$



With this, we can state the last property of intersection multiplicity.

Prop : $\mu_P(C,D) \geq \mathrm{mult}_P C \cdot \mathrm{mult}_P D$
with equality iff C,D have no tangent lines at P in common.

proof: suppose that everything is affine, that $P=(0,0)$, that $C=\{F=0\}$, $D=\{G=0\}$ have no common component at P and $m=\mathrm{mult}_P C$, $n=\mathrm{mult}_P D$. This means that in the ring $\mathcal{O}_P=\mathcal{O}_{\mathbb{A}^2,P}$ we have $F\in \mathfrak{m}^m$, $G\in\mathfrak{m}^n$.
Consider the surjective map

$$0\to \ker\pi \to \mathcal{O}_P/(F,G) \xrightarrow{\pi} \mathcal{O}_P/(F,G)+\mathfrak{m}^{m+n} \to 0$$

We claim that $\dim_{\mathbb{C}} \mathcal{O}_P/(F,G,\mathfrak{m}^{m+n}) \geq m\cdot n$.

Observe that $\mathcal{O}_P/(F,G)+\mathfrak{m}^{n+m} = \left(\mathbb{C}[X,Y]/(F,G)+\mathfrak{m}^{n+m}\right)_{\mathfrak{m}}$
$= \mathbb{C}[X,Y]/(F,G)+\mathfrak{m}^{n+n}$ since this ring is local

We now claim that we have an exact sequence of $\mathbb{C}$-algebras

$$0\to\ker\alpha\to \begin{matrix}\mathbb{C}[X,Y]/\mathfrak{m}^n\\ \oplus \\ \mathbb{C}[X,Y]/\mathfrak{m}^m\end{matrix} \xrightarrow{\alpha} \mathbb{C}[X,Y]/\mathfrak{m}^{n+m} \to \mathbb{C}[X,Y]/(F,G)+\mathfrak{m}^{n+n} \to 0$$

$$\begin{pmatrix} a \\ b\end{pmatrix} \longmapsto aF+bG$$

We need to prove that α surjects onto the ideal (F,G) in $\mathbb{C}[X,Y]/\mathfrak{m}^{n+m}$: let $aF+bG\in(F,G)$, then we can split a,b into two summands according to the degree

$$aF+bG = (a_{<n}+a_{\geq n})F + (b_{<m}+b_{\geq m})G$$
$$= a_{<n}\cdot F + b_{<m}G \in \mathrm{Im}\,\alpha$$

Hence we get

$$\mu_p(C,D) = \dim_{\mathbb{C}} \mathcal{O}_P/(F,G) = \dim_{\mathbb{C}} \mathcal{O}_P/(F,G)+m^{m+n} + \dim_{\mathbb{C}} \operatorname{Ker}\pi$$

$$= \dim_{\mathbb{C}} \mathbb{C}[X,Y]/(F,G)+m^{m+n} + \dim_{\mathbb{C}} \operatorname{Ker}\pi$$

$$= \dim \mathbb{C}[X,Y]/m^{m+n} - \dim \mathbb{C}[X,Y]/m^{n} + \dim \operatorname{Ker}\pi$$

$$- \dim \mathbb{C}[X,Y]/m^{m} + \dim \operatorname{Ker}\alpha + \dim \operatorname{Ker}\pi$$

$$= m\cdot n + \dim \operatorname{Ker}\alpha + \dim \operatorname{Ker}\pi$$

Hence $\mu_p(C,D) \geq m\cdot n$ with equality iff π, α are injective.
We claim that α is injective iff C,D have no common tangent line i.e. iff F_m, G_n are coprime.

Suppose first F_m, G_n coprime and let $a,b \in \mathbb{C}(X,Y)$ $\deg(a)<n$, $\deg(b)<m$ s.t. $\deg(aF+bG) \geq n+m$. Let a_*, b_* be the smallest degree terms of a,b. Then it must be $a_* F_m + b_* G_n = 0$ in $\mathbb{C}[X,Y]$ by degree reasons. Since F_m, G_n are coprime it must be $a_* = c\cdot G_n$, $b_* = -c\cdot F_m$ but by degree reason it must be $c=0$, so $a_* = b_* = 0$ and $a=b=0$ as well.

Conversely, suppose F_m, G_n are not coprime; $F_m = H\cdot F_m'$ $G_n = H\cdot G_n'$. Then $G_n'\cdot F_m - F_m'\cdot G_n = 0$ so that $(G_n', F_m') \in \operatorname{Ker}\alpha$ and $(G_n', F_m') \neq 0$.

To conclude we need to prove that if F_m, G_n are coprime then π is injective as well, meaning that

$m^{n+m} \subseteq (F,G)$ in $\mathcal{O}_{\mathbb{A}^2,P}$. We prove this in the following three steps:

(a) Consider the ideal $(F_m, G_n) \subseteq \mathbb{C}[X,Y]$. Since F_m, G_n are coprime, we have an exact sequence of $\mathbb{C}[X,Y]$-modules given by

$$0 \to \mathbb{C}[X,Y] \xrightarrow{\binom{G_n}{-F_m}} \mathbb{C}[X,Y] \oplus \mathbb{C}[X,Y] \xrightarrow{(F_m,G_n)} \mathbb{C}[X,Y] \to \mathbb{C}[X,Y]/(F_m,G_n) \to 0$$

Observe that the modules in this exact sequence are graded, and that the maps respect the grading, since F_m, G_n are homogeneous. This means that the exact sequence

$$0 \to \mathbb{C}[X,Y]_{d-n-m} \xrightarrow{\binom{G_n}{-F_m}} \mathbb{C}[X,Y]_{d-n} \oplus \mathbb{C}[X,Y]_{d-m} \xrightarrow{(F_m,G_n)} \mathbb{C}[X,Y]_d \to \mathbb{C}[X,Y]_d/(F_m,G_n)_d \to 0$$

is exact for all d. In particular, if $d \geqslant m+n$ we get that

$$\dim_{\mathbb{C}} \mathbb{C}[X,Y]_d/(F_m,G_n)_d = \dim \mathbb{C}[X,Y]_d - \dim \mathbb{C}[X,Y]_{d-n} - \dim \mathbb{C}[X,Y]_{d-m} + \dim \mathbb{C}[X,Y]_{d-n-m}$$

$$= d+1-(d-n+1)-(d-m+1)+d-n-m+1 = 0$$

so $\mathbb{C}[X,Y]_d = (F_m, G_n)_d \quad \forall d \geqslant m+n$

(b) Consider the ideal $\mathfrak{m}^{m+n}$; it is generated by all monomials $x^a y^b$ of degree $m+n$. These are elements of $\mathbb{C}[X,Y]_{m+n} = (F_m, G_n)_{m+n}$.

Hence, $\mathfrak{m}^{m+n}$ is generated by elements of the form $a \cdot F_m + b \cdot G_n$, a, b homogeneous, $\deg(a) = n, \deg(b) = m$.

By definition we have

$$F = F_m + F_{m+1} + \dots \qquad \text{so} \qquad F_m = F - F' \quad \text{with} \quad F' \in \mathfrak{m}^{m+1}$$
$$G = G_n + G_{n+1} + \dots \qquad\qquad G_n = G - G' \qquad\qquad G' \in \mathfrak{m}^{n+1}$$

Hence we can write

$$aF_m + bG_n = aF + bG - aF' - bG' \in (F+G) + \mathfrak{m}^{m+n+1}$$

This shows that $\mathfrak{m}^{m+n} = (F, G) + \mathfrak{m}^{m+n+1}$

(c) Consider now in the ring $\mathcal{O}_{\mathbb{A}^2, p}/(F,G)$ the ideal $I = \mathfrak{m}^{m+n}$. Point (b) tells us that $\mathfrak{m} I = I$ and since the ring is local with maximal ideal $\mathfrak{m}$, Nakayama's Lemma tells us that $I = 0$. This means that $\mathfrak{m}^{m+n} \subseteq (F, G)$ in $\mathcal{O}_{\mathbb{A}^2, p}$ which is what we wanted to prove. □