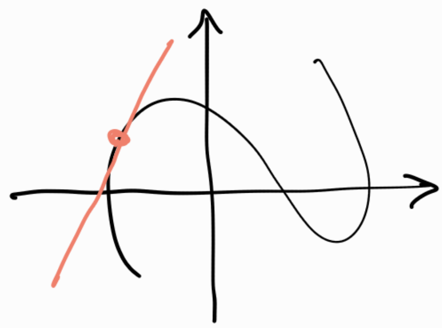
2.6. TANGENT SPACES and SINGULARITIES

While we were studying the intersection multiplicity we were looking at local properties of the intersection $C \cap D$. However we haven't yet studied the most important local property of a curve: the TANGENT SPACE.

We know what this should be intuitively:



However, we are going to give an unusual definition at first.

Let $C = \{F(x,y)=0\}$ be an affine plane curve and let $P \in C$ be a point.

Def: LOCAL RING of C

The local ring of C at P is

$$\mathcal{O}_{C,P} := \mathcal{O}_{\mathbb{A}^2,P} / (F)$$

This is a local ring where the unique maximal ideal is $m_P = (x-a, y-b)$ if $P=(a,b)$.

Def: TANGENT SPACE

The tangent space at P is $T_PC = \left(m_P / m_P^2 \right)^*$.

Let's look at this more closely. Suppose $P=(0,0)$ so that $m=(x,y)$. The general case can be brought to this by translations.

The vector space m_p/m_p^2 is generated as a vector space by x,y, because all the higher powers are zero. We have another relation which is

$$0 = F(x,y) = F_1(x,y) + \underbrace{F_2(x,y)+F_3(x,y)+\dots}_{\text{zero because we are modding out } m_p^2}$$
$$= F_1(x,y)$$

By Taylor's expansion we can write

$$F_1(x,y) = \frac{\partial F}{\partial x}(0)\cdot x + \frac{\partial F}{\partial y}(0)\cdot y$$

Hence

$$\boxed{m_0/m_0^2 \cong \frac{\mathbb{C}\cdot x \oplus \mathbb{C}\cdot y}{\left(\frac{\partial F}{\partial x}(0)\cdot x + \frac{\partial F}{\partial y}(0)\cdot y\right)}}$$

as $\mathbb{C}$-vector spaces. Dually we get

<u>Prop</u>: [JACOBI'S CRITERION]

The tangent space to C at p is identified with

$$\boxed{T_pC = \left\{(x,y)\in\mathbb{C}^2 \;\middle|\; \frac{\partial F}{\partial x}(P)\cdot x + \frac{\partial F}{\partial y}(P)\cdot y = 0\right\}}$$

Proof: Suppose $p=0$ as before. Thanks to the previous identification of m_0/m_0^2, an element of $T_0C = (m_0/m_0^2)^*$ is a $\mathbb{C}$-linear map

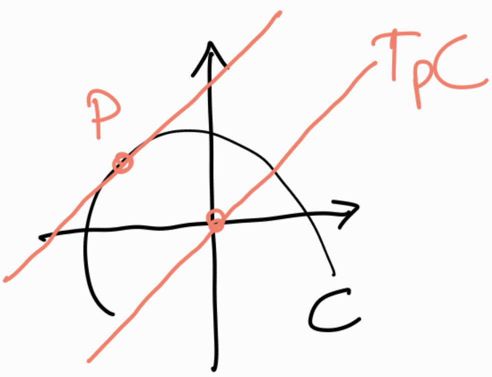
$$\varphi : \frac{\mathbb{C}\cdot x \oplus \mathbb{C}\cdot y}{\left(\frac{\partial F}{\partial x}(0)\cdot x + \frac{\partial F}{\partial y}(0)\cdot y\right)} \longrightarrow \mathbb{C}$$

which is the same as a linear map $\varphi : \mathbb{C}x \oplus \mathbb{C}y \to \mathbb{C}$ s.t. $\varphi\left(\frac{\partial F}{\partial x}(0)\cdot x + \frac{\partial F}{\partial y}(0)\cdot y\right) = 0$, which is the same as two elements $\varphi(x), \varphi(y) \in \mathbb{C}$ s.t.

$$\frac{\partial F}{\partial x}(0)\cdot \varphi(x) + \frac{\partial F}{\partial y}(0)\cdot \varphi(y) = 0$$

which is what we were aiming at. $\square$

Rmk: (1) The tangent space is a vector space by definition. If we translate to the point $p \in C$ we get the AFFINE TANGENT SPACE at p



If $C = \{F(X,Y,Z)=0\}$ is a projective plane curve and $P \in C$ is a point, we define its tangent space by restricting to an affine chart of $\mathbb{P}^2$.
In this affine chart we can take the affine tangent space and then take the Zariski closure in $\mathbb{P}^2$ and this is the PROJECTIVE TANGENT SPACE.

<u>Prop</u>: [PROJECTIVE JACOBI CRITERION]
Let $C = \{F(X,Y,Z)=0\}$ be a plane projective curve. The projective tangent space at $P \in C$ is

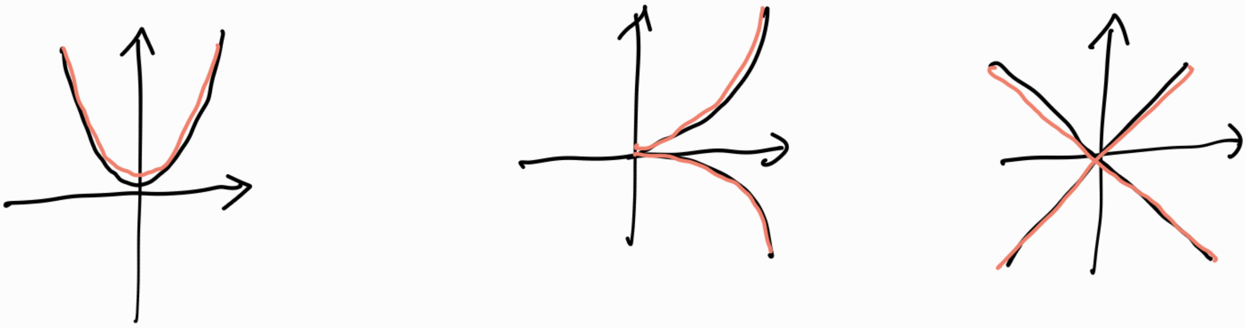
$$\mathbb{T}_P C = \left\{ [X,Y,Z] \;\middle|\; \frac{\partial F}{\partial X}(P)\cdot X + \frac{\partial F}{\partial Y}(P)\cdot Y + \frac{\partial F}{\partial Z}(P)\cdot Z = 0 \right\}$$

We expect the tangent space to a curve to be a line, this leads us to a fundamental definition:

<u>Def</u>: SMOOTH CURVE
A plane curve C (affine or projective) is called SMOOTH at a point P if $\dim_{\mathbb{C}} T_P C = 1$ and it is called SINGULAR otherwise.

A curve is called SMOOTH if it is smooth at every point. Is called singular otherwise.

<u>Example</u>: The curve $\{y - x^2 = 0\}$ is smooth at 0, with tangent line $\{y = 0\}$. The curves $\{y^2 - x^3 = 0\}$ and $\{x^2 - y^2 = 0\}$ are singular



Smooth curves are very geometric.

<u>Prop</u>: If a curve is smooth at P then C is reduced and irreducible at P.

<u>proof</u>: We can assume that the curve is affine and that $P = (0,0)$. Suppose then that $C = \{F = 0\}$ $= \{G \cdot H = 0\}$ where G, H are two factors of F, both vanishing at 0. Then $G, H \in m_P$ so $F \in m_P^2$. This means that the linear part of F is zero; $F_1 = 0$. So $m_P/m_P^2 = \mathbb{C}\cdot x \oplus \mathbb{C} y$.

Equivalently, we can use Leibniz' rule

$$\frac{\partial(G\cdot H)}{\partial x}(0) = \frac{\partial G}{\partial x}(0)\cdot H(0) + G(0)\cdot \frac{\partial H}{\partial y}(0) =$$

$$= \frac{\partial G}{\partial x}(0)\cdot 0 + 0 \cdot \frac{\partial H}{\partial y}(0) = 0. \qquad \square$$

Cor: Each smooth curve is reduced and any connected component is irreducible.

Proof: Let C be a smooth curve and write it as
$$C = n_1 C_1 + \dots + n_r C_r$$ with the C_i reduced and irreducible. At any point $p \in C_i$ the curve is reduced, hence $n_i = 1$, so $C = C_1 + \dots + C_r$.
The C_i must also be disjoint. Indeed if $p \in C_i \cap C_j$ then C is not smooth at p. Hence the irreducible components coincide with the connected components. $\square$

Rmk: (1) Since any smooth curve $C = \{F = 0\}$ is reduced, there is no difference between C and the set $\mathbb{V}(F)$.

As a consequence of Bezout we immediately get

Prop: Any smooth projective plane curve is reduced and irreducible.

proof: We know it is reduced. Suppose C has two distinct irreducible components C_1, C_2.
Then by Bezout they meet $C_1 \cap C_2 \neq \emptyset$. But then C is not smooth at $p \in C_1 \cap C_2$. $\square$

We can also connect the notion of tangent lines to the intersection multiplicity.

Let $C = \{F=0\}$ be a plane curve (affine or projective) and let $P \in C$ be a point.
Take also a line L (affine or projective), with $P \in L$
We can ask whether the line is contained in the tangent space or not.

Prop: L is contained in the tangent space at P if and only if $\mu_P(C, L) \geq 2$.

proof: suppose everything is affine and $P=(0,0)$.
Set $C = \{F(x,y)=0\}$, $L = \{\ell(x,y)=0\}$, we can also assume that $F(0)=\ell(0)=0$. Then L is tangent to C at P iff $\{\ell(x,y)=0\} \subseteq \{F_1(x,y)=0\}$ i.e. iff there exists $\lambda \in \mathbb{C}$ s.t. $F_1 = \lambda \cdot \ell$ i.e. iff $F_1(x,y), \ell(x,y)$ are linearly dependent. So we want

$$\mu_P(C,L) = 1 \iff F_1, \ell \text{ are linearly independent}$$

Suppose first that they are linearly independent. Then we can make a linear change of coordinates such that $F_1 = x$, $\ell = y$. Then

$$\mu_P(F, \ell) = \mu_P(x + \dots, y) = \mu_P(x + x^2 f(x), y) = 1$$

Conversely suppose that they are linearly dependent then we can make a linear change of coordinates and assume $\ell = y$, $F_1 = \lambda \cdot y$ for $\lambda \in \mathbb{C}$. Then

$$\mu_P(F, \ell) = \mu_P(\lambda \cdot y + F_2 + \dots, y) = \mu_P(F_2 + \dots, y)$$
$$= \mu_P(x^2 \cdot f(x), y) \geq 2. \qquad \square$$

This suggests to us the following: we say that two plane curves intersect TRANSVERSALLY at P if $\mu_P(C, D) = 1$. Then one can prove

<u>Prop</u>: Two curves C, D intersect transversally at P if and only if they are smooth at P with distinct tangent lines.

<u>Proof</u>: Suppose everything is affine, $P = (0\ 0)$ and $C = \{F = 0\}$, $D = \{G = 0\}$ with $F(0) = G(0) = 0$. Need:

$\mu_P(C, D) = 1$ iff $F_1(x, y), G_1(x, y)$ are linearly independent linear forms.

To prove this, let us look at our algorithm: each time we do step (b) we do not change the intersection number, nor F_1, G_1 being linearly independent.
Hence, we can look at the first time we go through step (a):

$$\mu_P(F, G) = \mu_P(F, y \cdot G') = \mu_P(F, y) + \mu_P(F, G')$$

since $\mu_P(F, y) \geq 1$ we have that $\mu_P(F, G) = 0$ iff $\mu_P(F, y) = 1$ and $\mu_P(F, G') = 0$ and since $\{y = 0\}$

is a line we know that this is equivalent to

F_1, γ linearly independent and $G'(0) \neq 0$

but this is like saying that F_1, G_1 are linearly independent. $\square$