We have seen last time the definition of intersection multiplicity of two affine plane curves, and we have seen how to compute the intersection multiplicity with a line, especially the x-axis: $\{y=0\}$.

[We follow here GATHMANN "Plane Algebraic Curves"]

In particular:

$$\mu_0(F(x,y),y) = \text{multiplicity of the root } x=0 \text{ in the polynomial } F(x,0)$$

__Example__: (1) $\mu_0(x^2+y^2+1, y) = 0$.

$$\mu_0(x+x^2+y^3, y) = \mu_0(x(x+1), y) = 1.$$

$$\mu_0(y-x^2, y) = \mu_0(-x^2, y) = 2.$$

Now we want to use this to compute the intersection multiplicity between any two curves.

Up to a translation, we can always suppose the point P to be $P=(0,0)$.

__Algorithm__: INTERSECTION MULTIPLICITY at 0

__Input__: two polynomials $F(x,y), G(x,y)$ with no irreducible factors in common that vanish at 0.

__Procedure__: There are three steps (a), (b), (c):

(a) If one of F, G has degree zero, then it is constant, so $\mu_0(F,G)=0$.

(2) If one of the curves contains the x-axis as a component, up to switching them, we can suppose that it is the curve D.

Hence $G(x,y) = y \cdot G'(x,y)$. So

$$\mu_0(F,G) = \mu_0(F,y) + \mu_0(F,G')$$

The first summand we can compute and in the second summand, $\deg G' < \deg G$. Observe that F and G' still have no factors in common.

(5) If none of the curves contains the x-axis, then, up to switching and multiplying by a nonzero scalar, we can suppose

$$F = x^n + \text{other terms only in } x \text{ with smaller degree} + \text{terms with } y$$
$$G = x^m + \text{other terms only in } x \text{ with smaller degree} + \text{terms with } y$$

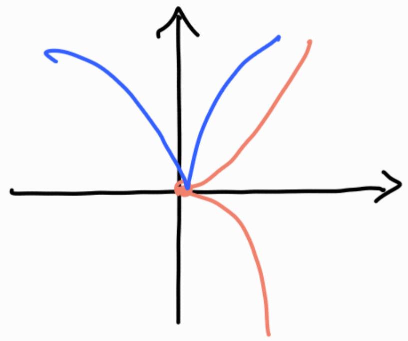
with $n \leq m$. Then

$$\mu_0(F,G) = \mu_0(F, G - x^{m-n} \cdot F)$$

Observe that F and $\cancel{G'}\ G - x^{m-n} \cdot F$ still have no factors in common.

Repeat these steps until we obtain the final intersection number.

Example: (1) $\mu_0(x^2-y^3,\ y^2-x^3)$



$= \mu_0(x^2-y^3,\ y^2-x^3+x\cdot(x^2-y^3))$

$= \mu_0(x^2-y^3,\ y^2-x^3+x^3-xy^3)$

$= \mu_0(x^2-y^3,\ y^2-xy^3)$

$= \mu_0(x^2-y^3,\ y^2)+\mu_0(x^2-y^3,\ 1-xy)$

$= 2\mu_0(x^2-y^3,\ y)+0 = 2\cdot 2+0 = 4.$

(2) $\mu_0(x+y^2,\ x+y^2-x^3)$

$= \mu_0(x+y^2,\ x+y^2-x^3+x^2\cdot(x+y))$

$= \mu_0(x+y^2,\ x+y^2+x^2y)$

$= \mu_0(x+y^2,\ x+y^2+x^2y-(x+y^2))$

$= \mu_0(x+y^2,\ y^2+x^2y-y^2) = \mu_0(x+y^2,\ x^2y)$

$= 2\mu_0(x+y^2,\ x)+\mu_0(x+y^2,\ y)$

$= 2\mu_0(y^2,x)+\mu_0(x,y) = 2\cdot 2+1 = 5$

Prop: The algorithm terminates and outputs the intersection multiplicity at 0.

proof: if it terminates, then it outputs the correct multiplicity. To see that it terminates consider the ideal (F,G). At each step of (b), we replace G with $G' = G - x^{m-n}F$ so the ideal (F,G) stays the same. However, when

we go to step (a), we have $G = y \cdot G'$
and we claim that either $F(0) \neq 0$ so $\mu_0(F,G) = 0$
or $F(0) = 0$ and $(F,G) \subsetneq (F,G')$

Indeed we have the exact sequence

$$0 \to \mathcal{O}_P/(F,y) \xrightarrow{\cdot G} \mathcal{O}_P/(F,G) \xrightarrow{\pi} \mathcal{O}_P/(F,G') \to 0$$

and if $F(0) = 0$ and $(F,G) = (F,G')$ then
we have that π is an isomorphism, so $\mathcal{O}_P/(F,y) = 0$
but then $\mu_0(F,y) = 0$, which is impossible
because both F, y vanish at 0.

Hence at each time we go through step (a), we
either conclude or we have a strict containment
$(F,G) \subsetneq (F,G')$. Since the ring $\mathbb{C}[x,y]$ is
Noetherian, we do not have infinite strictly
ascending chains of ideals. Hence we cannot
do step (b) infinitely many times. Since we
cannot do step (a) infinitely many times either
we must finish. $\square$

Cor: The intersection multiplicity of two curves
with no common components at P is finite.

Now we go towards Bezout's Theorem
First, we extend the definition of intersection multiplicity
to projective curves.

First, let $C = \{F(X,Y,Z)=0\}$ be a plane projective curve. This induces a plane affine curve in the open subset $\{Z \neq 0\} \cong \mathbb{A}^2$

$$C \cap \{Z \neq 0\} = \{ F(X,Y,1) = 0 \}$$

The analogous procedure holds for the two other subsets $\{X \neq 0\}$, $\{Y \neq 0\}$

<u>Def</u>: INTERSECTION MULTIPLICITY of PROJECTIVE CURVES

Let $P = [a,b,1] \in \mathbb{P}^2$ be a point and let $C = \{F(X,Y,Z)=0\}$, $D = \{G(X,Y,Z)=0\}$ be two plane projective curves. We define their intersection multiplicity at P as that of the corresponding plane affine curves in $\{Z \neq 0\}$.

$$\mu_P(C,D) = \mu_{(a,b)}\big(F(X,Y,1), G(X,Y,1) \big)$$

<u>Rmk</u>: (1) We can also use the corresponding affine curves in any affine set $\{X \neq 0\}$ $\{Y \neq 0\}$ or $\{Z \neq 0\}$ that contains P, the result does not change (<u>Exercise</u>: why?)

(2) The intersection multiplicity does not change under a linear change of coordinates $A: \mathbb{P}^2 \to \mathbb{P}^2$, $[x] \mapsto [Ax]$, $A \in GL(3,\mathbb{C})$.

Now we can finally state

Thm: BEZOUT'S THEOREM

Let C, D be two plane projective curves with no components in common. Then the curves intersect in $\deg D \cdot \deg C$ points, counted with multiplicity. More precisely:

$$\sum_{P \in \mathbb{P}^2} \mu_P(C,D) = (\deg C) \cdot (\deg D)$$

Lemma: Let $C = \{F=0\}$ and $D = \{G=0\}$ be two complex affine plane curves with no common components. Then the natural map

$$\mathbb{C}[x,y]/(F,G) \longrightarrow \prod_{P \in C \cap D} \mathcal{O}_P/(F,G)$$

is an isomorphism of $\mathbb{C}$-algebras.

proof: consider the $\mathbb{C}$-algebra $A = \mathbb{C}[x,y]/(F,G)$. We claim that all the prime ideals of A are those of the form $m_P = (x-a, y-b)$ where $P = (a,b) \in C \cap D$. Indeed, the prime ideals are the primes $\mathfrak{p}$ in $\mathbb{C}[x,y]$ s.t. $\mathfrak{p} \supseteq (F,G)$, so that $\mathbb{V}(\mathfrak{p}) \subseteq \mathbb{V}(F,G) = \{P_1, \dots, P_m\}$. Hence $\mathbb{V}(\mathfrak{p}) = \{P_i\}$ and $\mathfrak{p} = \mathbb{I}(\mathbb{V}(\mathfrak{p})) = \mathbb{I}(P_i) = m_{P_i}$. In particular, all the prime ideals are also maximal (i.e. A is Artinian)

We can then rephrase the Lemma by saying that the natural map

$$A \longrightarrow \prod_{\substack{\mathfrak{p} \subseteq A \\ \text{prime}}} A_{\mathfrak{p}} \quad \text{is an isomorphism}$$

This follows from a general fact about Artinian rings but we can give a proof as follows. Let $m_1, \dots, m_n$ be all the primes. Then they are coprime, because they are maximal, so that

$$m_1 \cdot \ldots \cdot m_n = m_1 \cap \ldots \cap m_n = \mathcal{N}(A)$$

where $\mathcal{N}(A)$ is the nilradical of A. We know that there exists $k > 0$ s.t. $\mathcal{N}(A)^k = 0$, so

$$0 = (m_1 \cdot \ldots \cdot m_n)^k = m_1^k \cdot \ldots \cdot m_n^k$$

and the m_i^k are also coprime, since the m_i are. Hence the Chinese Remainder Theorem shows that

$$A \cong A/_0 \cong A/m_1^k \cdot \ldots \cdot m_n^k = A/m_1^k \times \ldots \times A/m_n^k$$

and then it is easy to see that

$$A_{m_i} \cong (A/m_i^k)_{m_i} \cong A/m_i^k$$

which concludes the proof. $\square$

Thanks to this Lemma, we know that

$$\sum_{P \in C \cap D} \mu_P(C,D) = \dim \mathbb{C}[x,y]/(F,G)$$

and we need to compute the dimension of the latter

Lemma: Write F, G into homogeneous components

$$F = F_n + F_{n-1} + \dots + F_1 + F_0$$
$$G = G_m + G_{m-1} + \dots + G_1 + G_0$$

and assume that F, G have no common component and that F_n, G_m have no common component as well. Then

$$\dim_{\mathbb{C}} \mathbb{C}[x,y]/(F,G) = n \cdot m$$

proof: we want to describe $\mathbb{C}[x,y]/(F,G)$ in terms of the polynomial ring $\mathbb{C}[x,y]$. More precisely, consider the ideal (F,G) as a $\mathbb{C}[x,y]$-module. Since (F,G) is generated by two elements, we have a surjective map

$$\mathbb{C}[x,y]^{\oplus 2} \longrightarrow (F,G)$$
$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto aF + bG$$

What is the kernel of this map? It is the set of $\begin{pmatrix} a \\ b \end{pmatrix}$ s.t. $aF + bG = 0$, i.e. $aF = -bG$. Since F, G are coprime, it must be that $G \mid a$ and $F \mid b$, i.e. $a = k \cdot G$ and $b = h \cdot F$ and moreover

$$aF = -bG \iff kFG = -hFG \iff h = -k$$

So we have an exact sequence of $\mathbb{C}[x,y]$-modules

$$0 \to \mathbb{C}[x,y] \to \mathbb{C}[x,y]^{\oplus 2} \to (F,G) \to 0$$
$$k \longmapsto \begin{pmatrix} kG \\ -kF \end{pmatrix}, \quad \begin{pmatrix} a \\ b \end{pmatrix} \longmapsto aF + bG$$

which yields another exact sequence

$$0 \to \mathbb{C}[X,Y] \xrightarrow{\binom{G}{-F}} \begin{matrix}\mathbb{C}[X,Y] \\ \oplus \\ \mathbb{C}[X,Y]\end{matrix} \xrightarrow{(F,G)} \mathbb{C}[X,Y] \to \mathbb{C}[X,Y]/(F,G) \to 0$$

Now, to compute the dimension of $\mathbb{C}[X,Y]/(F,G)$ we could use the alternating sum of the other dimensions, but this does not make sense because the other dimensions are infinite. However, we can restrict ourselves to polynomials of bounded degrees. Fix $d \geq m+n$, then we claim that we have an exact sequence (*)

$$0 \to \mathbb{C}[X,Y]_{\leq d-n-m} \xrightarrow{\binom{G}{-F}} \begin{matrix}\mathbb{C}[X,Y]_{\leq d-n} \\ \oplus \\ \mathbb{C}[X,Y]_{\leq d-m}\end{matrix} \xrightarrow{(F,G)} \mathbb{C}[X,Y]_{\leq d} \xrightarrow{\pi} \mathbb{C}[X,Y]/(F,G)$$

To do this, it is enough to prove that any polynomial $f \in (F,G)$ of degree $\leq d$ can be written as $f = a \cdot F + b \cdot G$ with $\deg(a) \leq d-n$, $\deg(b) \leq d-m$. Write $f = a \cdot F + b \cdot G$ where a has the minimal possible degree. Then suppose $\deg a > d-n$ or $\deg b > d-m$. This means that aF or bG have degree larger than d and since $f = aF + bG$ has degree $\leq d$, the top degree terms must cancel: let a_*, b_* be the top degree terms of a, b respectively. Then $a_* F_n + b_* G_m = 0$ and since F_n, G_m are coprime, there must be a polynomial c such that $a_* = c \cdot G_m$, $b_* = -c \cdot F_n$.

Now we see that

$$f = aF + bG = (a - c\cdot G)\cdot F + (b + c\cdot F)\cdot G$$

however, $a - cG$ has degree strictly smaller than a, a contradiction, since we choose a of minimal degree. So we have the exact sequence $(*)$ of finite dimensional vector spaces and we can compute

$$\begin{aligned}
\dim \operatorname{Im} \pi &= \dim \mathbb{C}[x,y]_{\leq d} \\
&\quad - \dim \mathbb{C}[x,y]_{\leq d-m} - \dim \mathbb{C}[x,y]_{\leq d-n} \\
&\quad + \dim \mathbb{C}[x,y]_{\leq d-m-n} \\
&= \binom{d+2}{2} - \binom{d-m+2}{2} - \binom{d-n+2}{2} + \binom{d-m-n+2}{2} \\
&= \frac{1}{2}\Big[(d+2)(d+1) - (d-m+2)(d-m+1) - (d-n+2)(d-n+1) \\
&\qquad + (d-m-n+2)(d-m-n+1)\Big] \\
&= \frac{1}{2}\Big[\cancel{d^2} + 3d + 2 - \cancel{d^2} - (-m+2-m+1)d - (-m+2)(-m+1) \\
&\qquad - \cancel{d^2} - (-n+2-n+1)d - (-n+2)(-n+1) \\
&\qquad + \cancel{d^2} + (-m-n+2-m-n+1)d + (-m-n+2)(-m-n+1)\Big] \\
&= \frac{1}{2}\Big[(3+2m-3+2n-3-2m-2n+3)d + 2 \\
&\qquad - (m^2-3m+2) - (n^2-3n+2) \\
&\qquad + (m^2+m\cdot n-m+n\cdot m+n^2-n-2m-2n+2)\Big] \\
&= \frac{1}{2}\left[2\cdot mn\right] = m\cdot n.
\end{aligned}$$

$\square$