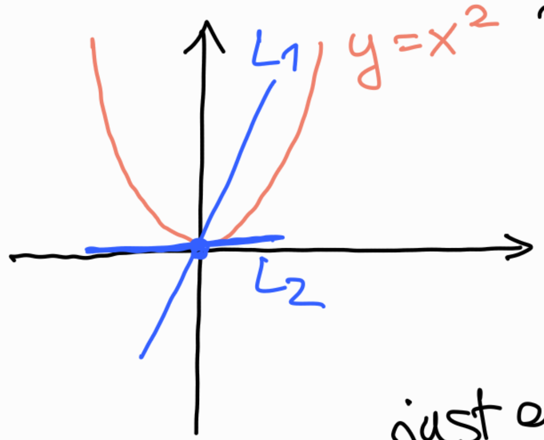
2.5 INTERSECTION MULTIPLICITY

We start with a spoiler. The result we are aiming at is

Thm [BEZOUT]
Two projective plane curves C, D with no components in common intersect in $\deg C \cdot \deg D$ points, counted with multiplicity.

We should define what we mean by counting points with multiplicity. This will be our task today.

Let's look for example at the case of a curve and a line



The affine curve $C = \{y = x^2\}$ intersects the line $L_1 = \{y = x\}$ transversally at 0, so that our intuition tells us that this is just a simple point of intersection.
The same intuition tells us that C intersects the line $L_2 = \{y = 0\}$ not transversally at 0 hence this should be more than a simple point of intersection. This can be made more precise as follows:

L_1 : parametrized by $\{(\lambda, \lambda) \mid \lambda \in \mathbb{C}\}$ $\quad y - x^2 \rightsquigarrow \lambda - \lambda^2 = \lambda(1 - \lambda)$ simple root $\lambda = 0$

L_2 : parametrized by $\{(\lambda, 0) \mid \lambda \in \mathbb{C}\}$ $\quad y - x^2 \rightsquigarrow -\lambda^2$ double root $\lambda = 0$

We want to generalize this picture to other curves, for which for example we do not have a parametrization.

The notion of intersection multiplicity is going to be local: it depends only on what happens in the neighborhood of a point. This means that we can restrict to the case of affine curves.

Let's take a point $P \in \mathbb{A}^2$. Keeping the dictionary between algebra and geometry in mind, we need an algebraic object that describes the local situation around P.

<u>Def</u>: LOCAL RING at P

Consider the maximal ideal $m_P \subseteq \mathbb{C}[x,y]$ corresponding to polynomials vanishing at P. The local ring at P is the localization

$$\mathcal{O}_P = \mathcal{O}_{\mathbb{A}^2,P} := \mathbb{C}[x,y]_{m_P} = \left\{ \frac{f}{g} \;\middle|\; \begin{matrix} f,g \in \mathbb{C}[x,y] \\ g(P) \neq 0 \end{matrix} \right\}$$

<u>Rmk</u>: (1) This is exactly the ring of rational function that are defined around P.

(2) We keep denoting the image of the ideal m_P, in $\mathcal{O}_P$ by m_P. This is the unique maximal ideal of $\mathcal{O}_P$.

Now, let $C = \{F=0\}$ and $D = \{G=0\}$ be two plane affine curve. We consider the ideal (F,G) and its image in $\mathcal{O}_P$ that we denote in the same way.

Def: INTERSECTION MULTIPLICITY

Let C, D be two plane curves as above and $P \in \mathbb{A}^2$ a point. We define the intersection multiplicity of C and D at P, or of F and G at P as

$$\boxed{\mu_P(C,D) = \mu_P(F,G) := \dim_{\mathbb{C}} \mathcal{O}_P/(F,G)}$$

This is a rather abstract definition. Let's try to understand some of its properties ← Example with coordinate axes

Properties of Intersection Multiplicity

(1) Consider an affine change of coordinates:

$$f: \mathbb{A}^2 \longrightarrow \mathbb{A}^2 \quad x \mapsto Ax+b \quad A \in GL_2$$

This sends a curve $C = \{F(x)=0\}$ to the curve $f(C) = \{F(f^{-1}(x))=0\}$. Then

$$\boxed{\mu_P(C,D) = \mu_{f(P)}(f(C), f(D))}$$

proof: f induces an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[x,y] \to \mathbb{C}[x,y] \qquad x \longmapsto Ax+b$$

which sends $m_{f(P)}$ to m_P

and $F(f^{-1}(x)), G(f^{-1}(x))$ to $F(x), G(x)$.

Hence it also gives an isomorphism

$$\mathcal{O}_{f(P)}/(F(f^{-1}(x)), G(f^{-1}(x))) \xrightarrow{\sim} \mathcal{O}_P/(F,G)$$

$\square$

(2) The intersection multiplicity is symmetric

$$\boxed{\mu_P(C,D) = \mu_P(D,C)}$$

proof: $(F,G) = (G,F)$. $\square$

(3) Let F, G, H be polynomials. Then

$$\boxed{\mu_P(F,G) = \mu_P(F, G + F\cdot H)}$$

proof: $(F,G) = (F, G+F\cdot H)$. $\square$

(4) $\boxed{\mu_P(C,D) \neq 0 \text{ iff } P \in C \cap D}$

proof: $\mu_P(C,D) = 0$ iff $(F,G) = \mathcal{O}_P$ i.e. iff $1 \in (F,G)$. Suppose that this happens, then there are rational functions $a, b \in \mathcal{O}_P$ s.t. $a\cdot F + b\cdot G = 1$. But then, evaluating at P we get $a(P)F(P) + b(P)G(P) = 1$. Hence it cannot be that $F(P) = G(P) = 0$. Conversely suppose that $P \notin C \cap D$, then, up to renaming, $F(P) \neq 0$. But then F is invertible in $\mathcal{O}_P$, so $(F,G) = \mathcal{O}_P$. $\square$

(5) If C, D have a component in common at P, then $\mu_P(C,D) = +\infty$.

proof: Exercise.

(6) The intersection multiplicity is bilinear:
if $C=\{F=0\}$, $D=\{G=0\}$, $E=\{H=0\}$ are plane curves

$$\boxed{\mu_p(C,D+E)=\mu_p(C,D)+\mu_p(C,E)}$$

proof: first suppose that C has a component through p in common with either D or E. Then both sides are equal to ∞ by (5). So we can assume that C has no component in common with either D or E. Then we can assume that F has no factors in common with G and H, because those factors that vanish at p are invertible in $\mathcal{O}_p$.
Now we claim that we have a short exact sequence

$$0\to \mathcal{O}_p/(F,G)\xrightarrow{\cdot H}\mathcal{O}_p/(F,G\cdot H)\xrightarrow{\pi}\mathcal{O}_p/(F,H)\to 0$$

of $\mathbb{C}$-vector spaces. Indeed the map $\mathcal{O}_p/(F,G\cdot H)\xrightarrow{\pi}\mathcal{O}_p/(F,H)$ is clearly surjective, and the kernel is given by

$$(F,H)/(F,G\cdot H)=\{aF+bH\}/(F,G\cdot H)=\frac{\{b\cdot H\mid b\in\mathcal{O}_p\}}{(F,G\cdot H)}$$

This is precisely the image of the map $\cdot H$.
We just need to show that this map is injective: suppose $b\cdot H=0$ in $\mathcal{O}_p/(F,GH)$. Then $b\cdot H=a\cdot F+c\cdot G\cdot H$ in $\mathcal{O}_p$ and multiplying by the denominators of a,b,c we get $A,B,C\in\mathbb{C}[x,y]$ s.t. $B\cdot H=A\cdot F+C\cdot GH$ in $\mathbb{C}[x,y]$. But since H has no component in common with F, it must be that $H\mid A$, so $A=A'\cdot H$. So we can write

$B = A' \cdot F + C \cdot G$, but then B and also b are zero in $\mathcal{O}_P/(F,G)$.

$\square$

Example : (1) Intersection with one line

We want to check that this definition of intersection multiplicity matches with our intuition, when L is a line.

Up to an affine change of coordinates, we can assume that $P=(0,0)$ and that $L = \{y=0\}$ is the x-axis. Let $C = \{f(x,y)=0\}$ be a curve which does not contain L as a component.

$$\begin{aligned}
\mu_0(y, f(x,y)) &= \mu_0(y,\ f(x,0) + y \cdot g(x,y)) \\
&= \mu_0(y,\ f(x,0)) \\
&= \mu_0(y,\ x^m \cdot h(x)) \quad , h(0) \neq 0 \\
&= \mu_0(y, x^m) + \mu_0(y, h(x)) \\
&= m \cdot \mu_0(y, x) + 0 \ = \ m \cdot 1 = m
\end{aligned}$$

So the intersection is exactly the multiplicity of the root in the polynomial obtained by putting a parametrization of L in the equation of C.