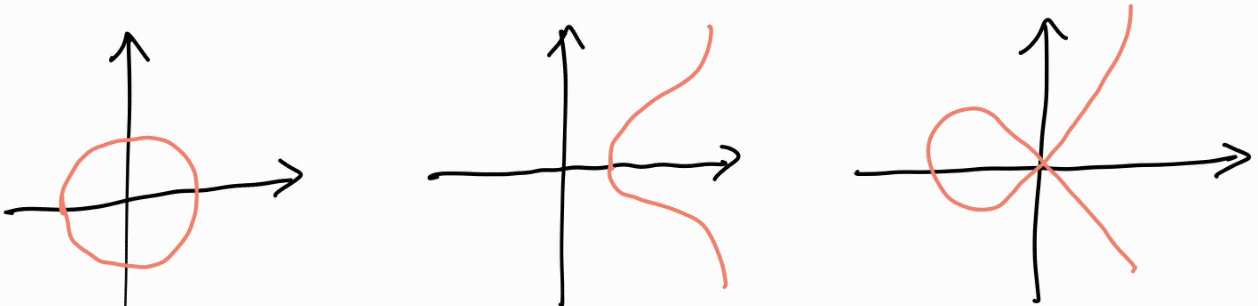
2.2: PLANE AFFINE CURVES

We want to start our exploration of curves from the most concrete case: plane curves.

It is easy to imagine what they are: subsets of $\mathbb{A}^2_{\mathbb{C}}$ described by one polynomial equation:

$$\mathbb{V}(f) = \{ f(x,y) = 0 \} = \{ \sum a_{ij} x^i y^j = 0 \}$$



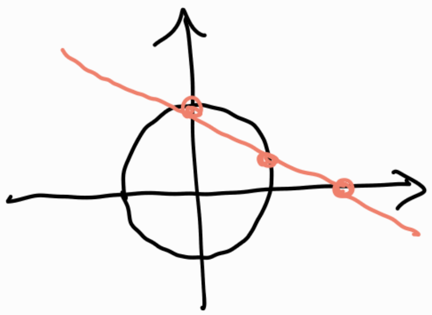
It is straightforward to find the irreducible components of such a curve: for $f \in k[x,y]$ let $f = f_1^{n_1} \cdot f_2^{n_2} \cdots f_m^{n_m}$ be a decomposition into irreducible factors. Then

Lemma: $\mathbb{V}(f) = \mathbb{V}(f_1) \cup \dots \cup \mathbb{V}(f_m)$ is the decomposition of $\mathbb{V}(f)$ into irred. comp.

proof: Since f_i is irreducible, the ideal (f_i) is prime. Hence $\mathbb{I}(\mathbb{V}(f_i)) = \sqrt{(f_i)} = (f_i)$ and $\mathbb{V}(f_i)$ is irreducible. Hence we have written

$\mathbb{V}(f) = \mathbb{V}(f_1) \cup \dots \cup \mathbb{V}(f_m)$ as a decomposition into irreducible closed subsets. Suppose $\mathbb{V}(f_i) \subseteq \mathbb{V}(f_j)$, then $\mathbb{I}(\mathbb{V}(f_i)) \supseteq \mathbb{I}(\mathbb{V}(f_j))$ so $(f_i) \supseteq (f_j)$, i.e. $f_i \mid f_j$. As they are irred. it must be $f_i = f_j$. $\square$

However, this is not going to be our definition of a plane curve. Why? One of the most important operations that we can do with plane curves is to intersect them. We have seen it already when we looked at linear projections:



The simplest case, that we have already considered is that of the intersection with a line:

$$\mathbb{V}(f) = \mathbb{V}\left(\sum a_{ij} x^i y^j \right)$$

$$\text{Line} = \mathbb{V}(aX + bY + c) = \left\{ \begin{pmatrix} a\cdot\lambda + b \\ c\cdot\lambda + d \end{pmatrix} \middle| \lambda \in \mathbb{C} \right\}$$

To compute the intersection we put the parametrization of the line into the equation of the curve

$$f(a\cdot\lambda + b, c\cdot\lambda + d) = 0$$

This is a polynomial in one variable λ, and its

roots correspond to the intersection points.
For example, we can know how many intersection points are there just by knowing the DEGREE of the polynomial $f(a\cdot\lambda+b, c\cdot\lambda+d)$, much easier than finding the roots explicitly.

However, this $\boxed{\text{DOES NOT ALWAYS WORK}}$, for the following simple reason: take the polynomial

$$f(x,y) = x^4-4x^3+2x^2y^2-8x^2y+12x^2 \\ -4xy^2+16xy-16x+y^4 \\ -8y^3+24y^2-32y+16$$

We want to intersect it with the x axis:

$$L=\{y=0\}=\{(\lambda,0) \mid \lambda\in\mathbb{C}\}.$$

By substituting, we get

$$f(\lambda,0)=\lambda^4-4\lambda^3+12\lambda^2-16\lambda+16$$

so we expect 4 roots and 4 intersection points.
However, if we actually compute the roots we get only two

$$\lambda=1\pm i\sqrt{3}$$

Indeed, it turns out that

$$f(\lambda,0)=(\lambda^2-2\lambda+4)^2$$

So there are two double roots. Maybe we could have avoided the problem by taking another line?

Actually, NO! Because it turns out that

$$f(x,y) = \left((x-1)^2+(y-2)^2-1\right)^2 = q(x,y)^2$$

so, any intersection with a line will give rise to

$$f(a\lambda+b, c\lambda+d) = q(a\cdot\lambda+b, c\cdot\lambda+d)^2$$

at least double roots. More generally, we have the same issue whenever f is not squarefree:

$$f = f_1^{n_1}\cdots f_m^{n_m} \quad \text{with one } n_i \geq 2$$

We could go around the issue by considering only squarefree polynomials, but this is hard: it is not obvious that the previous polynomial was not squarefree. More importantly, we want to keep the Fundamental Theorem of Algebra, saying that a polynomial $F(\lambda) \in \mathbb{C}[\lambda]$ has $\deg F$ roots, counted with multiplicity.

The solution to this is to consider also non-squarefree polynomials and to define an appropriate notion of multiplicity.

Thus we come to the definition:

Def: AFFINE PLANE CURVE

An affine plane curve is a polynomial $F \in k[X,Y]$, up to multiplication by a nonzero constant. Sometimes we write $C = \{F=0\}$.
The degree of a plane curve is the degree of the polynomial F.

Rmk: (1) For those who know about schemes, the affine plane curve $C=\{F=0\}$ is the scheme $\operatorname{Spec} k[X,Y]/(F)$ associated to F.

(2) To any plane curve $C = \{F=0\}$ we can associate a corresponding Zariski closed subset $\mathbb{V}(F) \subseteq \mathbb{A}^2$. However, two plane curves can yield the same closed subset
$$\mathbb{V}(F) = \mathbb{V}(F^2).$$

Def: REDUCED and IRREDUCIBLE PLANE CURVES

A plane curve $C=\{F=0\}$ is called reduced if the polynomial F is squarefree. It is called irreducible if $\mathbb{V}(F)$ is irreducible.

Rmk: (2) To any plane curve $C = \{F_1^{n_1} \cdots F_m^{n_m} = 0\}$ with the F_i irreducible we can associate the reduced plane curve $C^{red} = \{F_1 \cdots F_m = 0\}$.

(3) A plane curve $C = \{F=0\}$ is irreducible iff C^{red} is irreducible, it is reduced iff $C = C^{red}$.

Hence it is irreducible and reduced iff F is an irreducible polynomial.

Def: SUM of PLANE CURVES

Let $C = \{F=0\}$ and $D = \{G=0\}$ be two plane curves. Their sum is the plane curve

$$C+D = \{F \cdot G = 0\}$$

Rmk: (4) Any plane curve $C = \{F_1^{n_1} \cdot \ldots \cdot F_m^{n_m} = 0\}$ with the F_i irreducible can be written

$$C = n_1 C_1 + \ldots + n_m \cdot C_m$$

with the $C_i = \{F_i = 0\}$ irreducible and reduced

(5) We have that

$$\deg(C+D) = \deg(C) + \deg(D).$$

We introduced this definition of plane curves because we wanted to say something like

"If C is a plane curve of degree d and L is a line, then C and L intersect in d points, counted with multiplicity"

To make this statement precise, we need to define what we mean by multiplicity. However, we have a bigger problem: we can have two lines $L_1, L_2 \subseteq \mathbb{A}^2$ plane curves of degree 1, which are parallel, so that they do not meet at all! The solution to this will be to work in the PROJECTIVE PLANE $\mathbb{P}^2$.

2.3 : THE PROJECTIVE SPACE

The projective space $\mathbb{P}^n$ is a compactified version of the affine space $\mathbb{A}^n$. We can define it over an arbitrary field k.

Def : PROJECTIVE SPACE

The projective space $\mathbb{P}^n = \mathbb{P}^n(k)$ over a field k can be described as:

(a) The set of tuples $[a_0, \dots, a_n]$, $a_i \in k$ s.t. the a_i are not all zero and

$$\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix} \iff \exists \lambda \in k^* \text{ s.t. } \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = \lambda \cdot \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix} \text{ in } k^{n+1}.$$

(b) $\left(\mathbb{A}^{n+1} \setminus \{0\}\right) / k^*$ where k^* acts by scalar multiplication

(c) The set of lines in $\mathbb{A}^{n+1}$ passing through O.

We should check that these descriptions are equivalent: the equivalence between the first two descriptions is clear: they are the same, but one is expressed in symbols and the other one in words. About the equivalence with (c): a line in $\mathbb{A}^{n+1}$ passing through O is determined by any nonzero vector $\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} \neq 0$ on it.
Moreover, two nonzero vectors lie on the same line through O iff they differ by multiplication by a nonzero scalar.

We congeneralize this definition as follows:

<u>Def</u>: PROJECTIVE SPACE of a VECTORSPACE

Let V be a vector space over k. The corresponding projective space $\mathbb{P}(V)$ can be described as

(a) The set of nonzero vectors $[v]$, $v \in V, v \neq 0$ where $[v] = [w] \iff \exists \lambda \in k^*$ s.t. $v = \lambda \cdot w$.

(b) $(V \setminus \{0\})/k^*$ where k^* acts by scalar multiplication.

(c) The set of 1-dim'l subspaces of V.

<u>Rmk</u>: The first proj. space is a particular example of the second: $\mathbb{P}^n = \mathbb{P}(k^{n+1})$.

We have said already that $\mathbb{P}^n$ is a compactified version of $\mathbb{A}^n$. What do we mean by it?

$$\mathbb{P}^n = \{ [x_0, \dots, x_n] \mid (x_0, \dots, x_n) \neq 0 \}$$

$H_i = \{ x_i = 0 \}$ observe that this is well-defined: it is independent from the multiplication by a nonzero scalar.

$U_i = \{ x_i \neq 0 \}$ is like $\mathbb{A}^n$ via the mutually inverse maps

$$U_i \longrightarrow \mathbb{A}^n \qquad\qquad \mathbb{A}^n \longrightarrow U_i$$

$$\begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{pmatrix} x_0/x_i \\ x_1/x_i \\ \vdots \\ x_n/x_i \end{pmatrix} \qquad\qquad \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ 1 \\ \vdots \\ a_n \end{pmatrix} \text{ i-th position}$$

So we see that $\mathbb{P}^n = U_i \cup H_i \cong \mathbb{A}^n \cup \mathbb{P}^{n-1}$.

<u>Examples</u>: (1) $\mathbb{P}^1 = \{[x_0, x_1] \mid (x_0, x_1) \neq (0,0)\}$

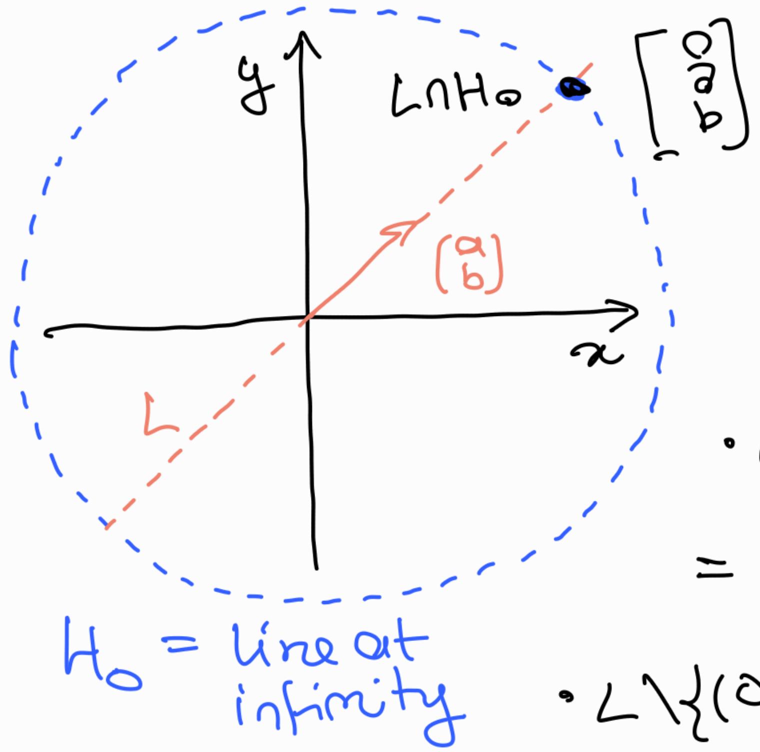
$H_0 = \{x_0 = 0\} = \{[0,1]\}$

$U_0 = \{x_0 \neq 0\} = \{[1,x] \mid x \in \mathbb{A}^1\}$

$\mathbb{P}^1 = \mathbb{A}^1 \cup \{[0,1]\} = \mathbb{A}^1 \cup \{\infty\}$.

(2) $\mathbb{P}^2 = \{[x_0, x_1, x_2]\}$, $U_0 = \{x_0 \neq 0\} = \{[1, x, y]\}$

$H_0 = \{x_0 = 0\} = \{[0, x_1, x_2]\}$



Consider in the affine space U_0 the line through the origin spanned by a nonzero vector $\begin{pmatrix} a \\ b \end{pmatrix}$:

- $L = \{(\lambda a, \lambda b) \mid \lambda \in \mathbb{C}\}$
$= \{[1, \lambda \cdot a, \lambda \cdot b] \mid \lambda \in \mathbb{C}\}$

- $L \setminus \{(0,0)\} = \{[1, \lambda a, \lambda b] \mid \lambda \in \mathbb{C}^*\}$

$= \{[\lambda^{-1}, a, b] \mid \lambda \in \mathbb{C}^*\} \xrightarrow[\lambda \to +\infty]{} [0, a, b]$

This can be made precise.

The algebraic object corresponding to $\mathbb{P}^n$ is again the polynomial ring $k[X_0,..,X_n]$ but this time with a grading:

$F(X_0,..,X_n)$ homogeneous polynomial of degree d

$\mathbb{V}(F) \subseteq \mathbb{P}^n$: $\mathbb{V}(F) = \{ [X_0,..,X_n] \mid F(X_0,..,X_n) = 0 \}$

is well-defined: if $\lambda \in \mathbb{C}^*$

$$\boxed{F(\lambda X_0,.., \lambda X_n) = \lambda^d \cdot F(X_0,..,X_n)}$$ so LHS $=0$ iff RHS $=0$.

In general, we can define $\mathbb{V}(I) \subseteq \mathbb{P}^n$ whenever $I \subseteq k[X_0,..,X_n]$ is a homogeneous ideal: an ideal generated by homogeneous elements

$$\mathbb{V}(I) = \left\{ [X_0,..,X_n] \in \mathbb{P}^n \;\middle|\; F(X_0,..,X_n) = 0 \quad \begin{array}{l} \forall F \in I \\ F \text{ homogeneous} \end{array} \right\}$$

Def: ZARISKI TOPOLOGY

The topology on $\mathbb{P}^n$ where the closed subsets are those of the form $\mathbb{V}(I)$, for I homogeneous ideal.

If $X \subseteq \mathbb{P}^n$ is closed, we can define a homogeneous ideal

$\mathbb{I}(X) =$ homogeneous ideal generated by homogeneous polynomials F s.t. $F(p) = 0 \;\; \forall p \in X$

and we have a projective Nullstellensatz

Thm: PROJECTIVE NULLSTELLENSATZ

Let $J \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal where k is an alg. clos. field. Let $\mathbb{V}(J) \subseteq \mathbb{P}^n$

(a) $\mathbb{V}(J) = \emptyset \iff \sqrt{J} \supseteq (x_0, \dots, x_n)$

(b) If $\mathbb{V}(J) \neq \emptyset$ then $\mathbb{I}(\mathbb{V}(J)) = \sqrt{J}$.

And we have the previous dictionary

Algebra			Geometry	
Graded Poly. Ring	$k[x_0, \dots, x_n]$		Projective Space	$\mathbb{P}^n$
Homogeneous ideals	J		Zariski closed subsets	X
Radical hom. ideals	J	"$\longleftrightarrow$"	Zariski closed subsets	X
Prime hom. ideals	$\mathfrak{p}$	"$\longleftrightarrow$"	irreducible Zariski closed	X

Now we can start working with projective plane curves. Of course we could define them as

$$\mathbb{V}(F) \subseteq \mathbb{P}^2, \quad F \in \mathbb{C}[X, Y, Z] \text{ homogeneous}$$

but we are going to take a slightly different route.

2.4 : PROJECTIVE PLANE CURVES

Def: PROJECTIVE PLANE CURVE

A projective plane curve is an homogeneous polynomial $F \in \mathbb{C}[X,Y,Z]$ up to multiplication by a nonzero constant. We write $C = \{F = 0\}$. The DEGREE of the curve is the degree of F.

Rmk: (1) For those who know about schemes this is the subscheme defined by F: $\operatorname{Proj} \mathbb{C}[X,Y,Z]/(F)$.

(2) The space of plane curves of degree d is by definition $\mathbb{P}(\mathbb{C}[X,Y,Z]_d)$.

(3) All the properties and operations that we have defined for affine plane curves extend to the projective case: reduced and irreducible curves, sum of curves, ...

(4) Plane curves of degree ONE are called LINES

ONE	LINES
TWO	CONICS
THREE	CUBICS
FOUR	QUARTICS
⋮	⋮

One of the reasons we introduced the projective space is to go around the phenomenon of parallel lines.

<u>Prop</u>: Two distinct lines in $\mathbb{P}^2$ meet in exactly one pt.

<u>proof</u>: $L_1 = \{ a_1 X + b_1 Y + c_1 Z = 0 \}$
$L_2 = \{ a_2 X + b_2 Y + c_2 Z = 0 \}$

distinct. This means that $\begin{matrix}(a_1,b_1,c_1)\\(a_2,b_2,c_2)\end{matrix}$ are linearly independent, equivalently that the matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ has rank 2. Then

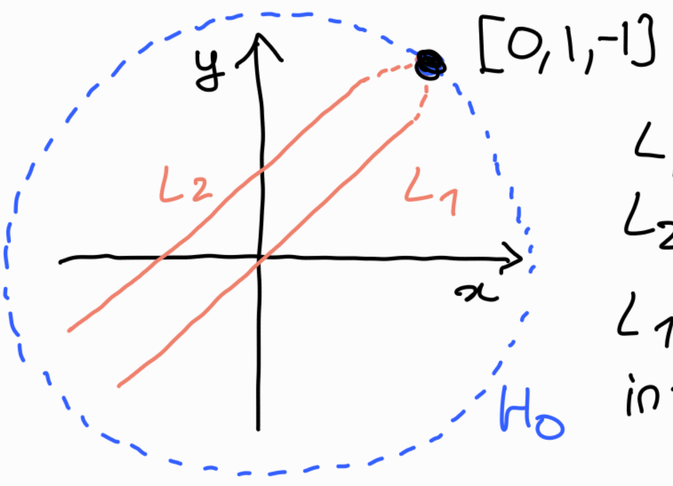
$$L_1 \cap L_2 = \begin{cases} a_1 X + b_1 Y + c_1 Z = 0 \\ a_2 X + b_2 Y + c_2 Z = 0 \end{cases}$$

$$\iff \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0 .$$

Since the matrix has rank two, there is a one dimensional space of solutions to that linear system. We can write any solution as $\lambda \cdot \begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix}$ where $(\bar{X}, \bar{Y}, \bar{Z})$ is one nonzero solution.

So we see that $L_1 \cap L_2 = \{ [\bar{X}, \bar{Y}, \bar{Z}] \}$ one point.

<u>Example</u>:



$L_1 = \{ y = x \}$
$L_2 = \{ y = x + 1 \}$
L_1, L_2 meet at infinity in the point $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.