# § 2. PLANE ALGEBRAIC CURVES

## 2.1 : IDEALS and VARIETIES

The objects at the center of our course are varieties: geometric objects defined by the vanishing of polynomial equations. There is a strong interplay between the geometry and the algebra, that we want to recall here.

Most of our results will be over the field of complex numbers $\mathbb{C}$, but many notions can be stated over an arbitrary field $k$.

| Algebra | Geometry |
|---|---|
| Polynomial ring: $k[X_1,\dots,X_n]$ | Affine space: $\mathbb{A}^n(k) = \mathbb{A}^n = k^n$ |
| Polynomials: $f_1,\dots,f_s$ | Algebraic set: $\mathbb{V}(f_1,\dots,f_s) = \{x \in \mathbb{A}^n \mid f_i(x) = 0\}$ |

Observe that if we pass from the polynomials $f_1,\dots,f_s$ to the ideal $I$ that they generate, the variety does not change

$$I = (f_1,\dots,f_s) = \left\{ \sum_{i=1}^{s} g_i f_i \mid g_i \in k[X_1,\dots,X_n] \right\}$$

$$\mathbb{V}(I) = \{x \in \mathbb{A}^n \mid \underset{\forall f \in I}{f(x) = 0}\} = \mathbb{V}(f_1,\dots,f_s)$$

Furthermore, any ideal in $k[x_1, \dots, x_n]$ is finitely generated:

<u>Thm</u> [HILBERT'S BASISSATZ]
The ring $k[x_1, \dots, x_n]$ is Noetherian, meaning that one of the following equivalent conditions hold

(a) Every ideal $I$ is finitely generated
(b) There is no infinite strictly increasing chain of ideals: $I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$

<u>Rmk</u>: This can be made concrete in terms of Groebner bases.

So we can update our dichotomy to

| Algebra | | Geometry | |
|---|---|---|---|
| Poly ring | : $k[x_1, \dots, x_n]$ | Affine space | : $\mathbb{A}^n$ |
| Ideal | : $I \subseteq k[x_1, \dots, x_n]$ | Algebraic set | : $\mathbb{V}(I)$ |

This correspondence satisfies some easy rules

- $\mathbb{V}(I_1 + I_2) = \mathbb{V}(I_1) \cap \mathbb{V}(I_2)$
- $\mathbb{V}(\sum I_j) = \bigcap \mathbb{V}(I_j)$
- $\mathbb{V}(I_1 \cap I_2) = \mathbb{V}(I_1 \cdot I_2) = \mathbb{V}(I_1) \cup \mathbb{V}(I_2)$
- if $I_1 \subseteq I_2$ then $\mathbb{V}(I_1) \supseteq \mathbb{V}(I_2)$
- $\mathbb{V}(\sqrt{I}) = \mathbb{V}(I)$

In particular, algebraic sets of the form $\mathbb{V}(I)$ are closed under finite union, infinite intersection and furthermore $\mathbb{A}^n = \mathbb{V}(0)$, $\phi = \mathbb{V}(1)$. Hence they form the closed subsets of a topology

<u>Def</u>: ZARISKI TOPOLOGY

This is the topology on $\mathbb{A}^n$ where the closed subsets are those of the form $\mathbb{V}(I)$.

<u>Example</u>: (1) The closed subsets of $\mathbb{A}^1$ are precisely the whole $\mathbb{A}^1$ and all the finite subsets.

<u>proof</u>: Suppose that $I$ has a nonzero polynomial $f(x) \in I$. Then $\mathbb{V}(I) \subseteq \mathbb{V}(f)$ and the second is finite with at most $\deg(f)$ elements. Conversely if $\{a_1, .., a_n\} \subseteq \mathbb{A}^1$ is finite, then $\{a_1, .., a_n\} = \mathbb{V}((x-a_1) \cdots (x-a_n))$ is closed. $\square$

We have seen how to obtain sets from ideals. Conversely from a set $X \subseteq \mathbb{A}^n$ we can define its vanishing ideal as $\mathbb{I}(X) = \{ f \in k[x_1, .., x_n] \mid f(p) = 0 \ \forall p \in X \}$

The fundamental relation between the two notions is given by the following fundamental theorem.

Thm: [HILBERT'S NULLSTELLENSATZ]
Let $k$ be an algebraically closed field and $I \subseteq k[X_1, \dots, X_n]$ an ideal. Then

$$\mathbb{I}(\mathbb{V}(J)) = \sqrt{J}$$

Rmk: (1) This is a wide generalization of the Fundamental Theorem of Algebra: let $f(x) \in k[x]$ be a polynomial with no roots. Then

$$\mathbb{I}(\mathbb{V}(f)) = \mathbb{I}(\emptyset) = k[X_1, \dots, X_n]$$

so that $\sqrt{(f)} = k[X_1, \dots, X_n]$. This means that $1 \in \sqrt{(f)}$ and so that $1 \in (f)$. But then $f$ must be a nonzero constant polynomial.

(2) The theorem fails if $k$ is not algebraically closed: over $\mathbb{R}$, $\mathbb{V}(x^2+y^2) = \{(0,0)\}$ so

$$\mathbb{I}(\mathbb{V}(x^2+y^2)) = \mathbb{I}((0,0)) = (x,y) \neq \sqrt{(x^2+y^2)}.$$

In general, it only holds that $\mathbb{I}(\mathbb{V}(J)) \supseteq \sqrt{J}$.

(3) In particular, we see that

$$\mathbb{I}(\mathbb{V}(J)) = J \iff J \text{ is radical}$$

We will not prove this result but the proof is on any good introductory book on algebraic geometry. Maybe we will discuss a proof in the exercises.

In the following, we will always assume $k =$ alg. closed

So we can update again our dictionary to

| Algebra | | Geometry |
|---|---|---|
| Polynomials $k[X_1, \dots, X_n]$ | | Affine Space $k^n$ |
| ideals $J$ | $\longrightarrow$ | closed subsets $V(J)$ |
| ideal $I(X)$ | $\longleftarrow$ | closed subsets $X$ |
| radical ideals $J$ | $\longleftrightarrow$ | closed subsets $X$ |

Let's expand this further: on the algebraic side we also have prime ideals. They correspond to irreducible sets on the geometric side.

**Def**: IRREDUCIBLE SET

A topological space $X$ is called irreducible if it cannot be written as $X = X_1 \cup X_2$ where $X_1, X_2$ are two distinct, nonempty closed subsets.

An irreducible component of $X$ is a closed irreducible subset which is not strictly contained in another closed irreducible subset.

**Rmk**: (1) A topological space $X$ is irreducible iff any nonempty open subset $U \subseteq X$ is dense.

proof: Let $U, V$ be open subsets of $X$. Then

$$U \cap V = \emptyset \iff (X \setminus U) \cup (X \setminus V) = X. \quad \square$$

(2) If $Y \subseteq X$ is an irreducible subspace, the closure $\overline{Y} \subseteq X$ is again irreducible.

proof: Suppose $\overline{Y} = Y_1 \cup Y_2$ with $Y_i$ closed in $\overline{Y}$. Then they are also closed in $X$, and $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$. Since $Y$ is irreducible, we can assume without loss of generality (w.l.o.g.) that $Y = Y \cap Y_1$, i.e. $Y \subseteq Y_1$. But then $\overline{Y} \subseteq Y_1$ and we are done. □

(3) Open subsets of irreducible spaces are connected.

proof: follows from (1).

(4) Irreducible spaces are connected.

Prop: A Zariski closed subset $X \subseteq \mathbb{A}^n$ is irreducible if and only if $\mathbb{I}(X)$ is a prime ideal.

Proof: Suppose $X$ is irreducible and let $f \cdot g \in \mathbb{I}(X)$. Then $\mathbb{V}(f \cdot g) \supseteq X$ so that $\mathbb{V}(f) \cup \mathbb{V}(g) \supseteq X$. Hence $X = (X \cap \mathbb{V}(f)) \cup (X \cap \mathbb{V}(g))$. Since $X$ is irreducible, one of these closed subsets coincides with $X$. Suppose $X = X \cap \mathbb{V}(f)$. Then $X \subseteq \mathbb{V}(f)$ i.e. $f \in \mathbb{I}(X)$.

Conversely, suppose that $\mathbb{I}(X)$ is prime and write $X = X_1 \cup X_2$ as union of two closed subsets. Then $\mathbb{I}(X) = \mathbb{I}(X_1) \cap \mathbb{I}(X_2)$. We want to show that one of $\mathbb{I}(X_1), \mathbb{I}(X_2)$ is contained in $\mathbb{I}(X)$. Suppose that $\mathbb{I}(X_2) \not\subseteq \mathbb{I}(X)$: then there is $g \in \mathbb{I}(X_2)$, $g \notin \mathbb{I}(X)$. But then for any $f \in \mathbb{I}(X_1)$ we have $f \cdot g \in \mathbb{I}(X_1) \cap \mathbb{I}(X_2) = \mathbb{I}(X) \Rightarrow f \in \mathbb{I}(X)$ because $\mathbb{I}(X)$ is prime.

**Prop:** Any Zariski closed set $X \subseteq \mathbb{A}^n$ has a finite number of irreducible components, and $X$ is the union of those components $X = X_1 \cup \dots \cup X_m$.

**proof:** algebraically, this is a byproduct of primary decomposition: given any radical ideal $J$ in $k[x_1, \dots, x_n]$ there are finitely many distinct prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ such that

$$J = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_m$$

and moreover there is no other prime ideal $\mathfrak{p}$ s.t. $J \subseteq \mathfrak{p} \subsetneq \mathfrak{p}_i$ for some $i$. Geometrically this translates to $\mathbb{V}(J) = \mathbb{V}(\mathfrak{p}_1) \cup \dots \cup \mathbb{V}(\mathfrak{p}_m)$ where the $\mathbb{V}(\mathfrak{p}_i)$ are distinct irreducible components of $\mathbb{V}(J)$.
There is also a geometric proof, that we might see in the exercises. $\square$

So we have another update to our dictionary

| Algebra | | | Geometry | |
|---|---|---|---|---|
| Polynomial ring | $k[x_1, \dots, x_n]$ | | Affine space | $\mathbb{A}^n$ |
| Ideals | $J$ | $\rightarrow$ | Zariski closed | $\mathbb{V}(J)$ |
| Ideals | $\mathbb{I}(X)$ | $\leftarrow$ | Zariski closed | $X$ |
| Radical ideals | | $\leftrightarrow$ | Zariski closed | |
| Prime ideals | | $\leftrightarrow$ | irreducible closed | |