ALGEBRAISCHE KURVEN
ALGEBRAIC CURVES

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LECTURES : Tuesday, 12 – 14 . Raum 5O9 (C6O5)
Thursday, 12 – 14 . Raum 5O9 (C6O5)

EXERCISES : Wednesday, 16-18. Raum C4H33.

EXAM : Oral exam. 60% points from the exercise sheets required.

URM : register at https://urm.math.uni-tuebingen.de

NOTES & EXERCISE SHEETS : on the webpage.

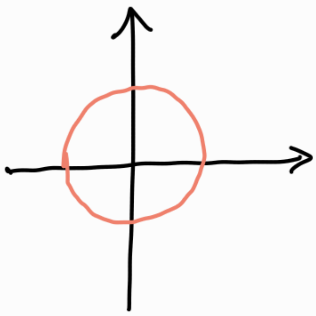
ONLINE LECTURES : we might need to hold some of the lectures / exercise classes online.

PREREQUISITES : basics of commutative algebra and complex analysis.

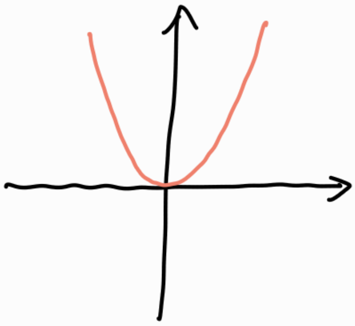
SEMINAR : there will be a related seminar in the SS 2023

§ 1. INTRODUCTION

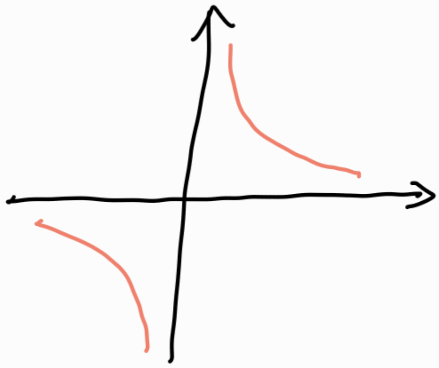
This course is about ALGEBRAIC CURVES. These are algebraic varieties: sets defined by the vanishing of polynomial equations



$x^2+y^2=1$

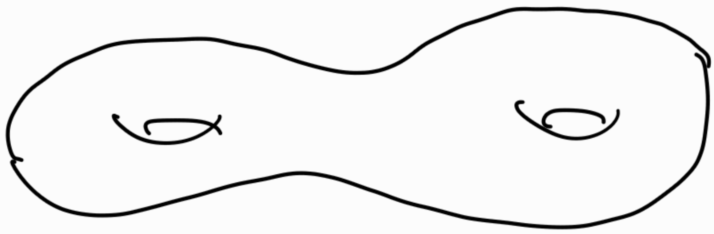


$y=x^2$



$xy-1=0$

At the same time, smooth algebraic curves are RIEMANN SURFACES: complex manifolds of dimension one



A compact Riemann surface of genus 2

Hence algebraic curves have at least a double life: algebraic and analytic. They also have other lives: arithmetic, in physics, ... In the course we will focus on the algebraic aspects with a touch of analytic flavour.

The theory of curves is one of the foundations of algebraic geometry and it is still extremely active.

1.1 INTEGRALS

A large part of the theory arose from the motivation of calculating integrals.

Say that we want to compute the integral

$$\int_a^b \frac{1}{(x-1)(x-2)(x-3)}dx \text{. How do we do it?}$$

Well, we can observe that

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{1}{2}\cdot\frac{1}{(x-1)} - \frac{1}{(x-2)} + \frac{1}{2}\frac{1}{(x-3)}$$

and then we can compute a primitive

$$\int \frac{1}{(x-1)(x-2)(x-3)}dx = \frac{1}{2}\int\frac{1}{x-1}dx - \int\frac{1}{x-2}dx + \frac{1}{2}\int\frac{1}{x-3}dx$$
$$= \frac{1}{2}\log(x-1) - \log(x-2) + \frac{1}{2}\log(x-3)$$

Let's try another one: we want to compute a primitive

$$\int \frac{1}{x^2+1}dx = \arctan(x) \quad \left[\text{if we remember our calculus, I don't}\right]$$

or we can write $\displaystyle \frac{1}{x^2+1} = \frac{1}{(x-i)(x+i)} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$

and then we have

$$\int \frac{1}{x^2+1}dx = \frac{1}{2i}\left[\log(x-i) - \log(x+i)\right]$$

these two answers are the same (up to a constant) and if one takes the correct branch of the logarithm.

In general, we can compute the integral of any RATIONAL FUNCTION : $\frac{p(x)}{q(x)}$ where $p(x), q(x)$ are two polynomials.

We can write, over the complex numbers $\mathbb{C}$

$$\frac{p(x)}{q(x)} = r(x) + \frac{a_1}{(x-b_1)^{k_1}} + \frac{a_2}{(x-b_2)^{k_2}} + \dots + \frac{a_m}{(x-b_m)^{k_m}}$$

for one polynomial $r(x) \in \mathbb{C}[x]$ and $a_i, b_i \in \mathbb{C}$, $k_i \in \mathbb{N}$.

We know how to integrate each of these separately.

So, we know how to compute integrals of the form

$$\int_a^b \frac{p(x)}{q(x)} dx$$ and we can do so rather algebraically.

Let's consider something different : we want to compute

$$\int \frac{1}{\sqrt{1-x^2}} dx$$

We have here at least two methods:

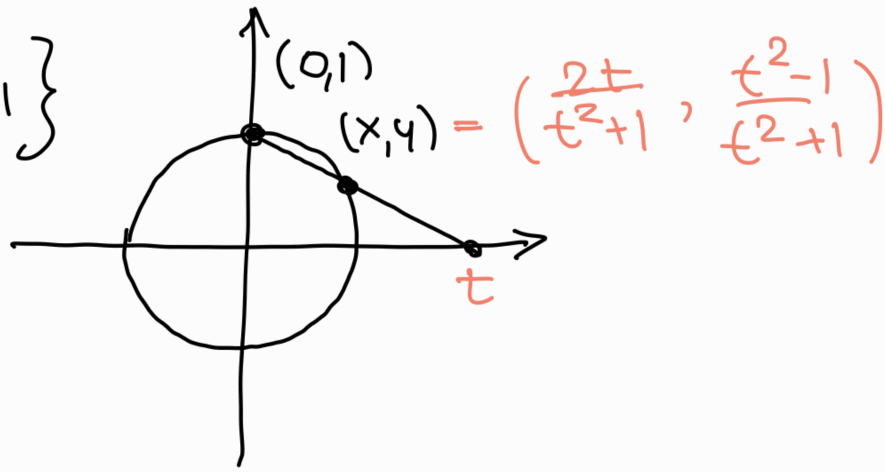
- CALCULUS: $x = \cos\theta \quad dx = -\sin\theta \, d\theta$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \int -\frac{1}{\sqrt{1-\cos^2\theta}} \sin\theta \, d\theta = \int \frac{\sin\theta}{\sqrt{\sin^2\theta}} d\theta = \int d\theta = \theta$$

- ALGEBRAIC GEOMETRY: we want to go back to the situation that we know how to solve: rational functions.

Set $y = \sqrt{1-x^2}$. Then $x^2+y^2 = 1$ and the integral becomes $\int \frac{1}{y}\,dx$. Now, let's look at the curve $x^2+y^2=1$. This is the familiar circle

$$C = \{x^2+y^2=1\}$$



This curve has a rational parametrization: a parametrization $x = f(t)$, $y = g(t)$ by two rational functions $f(t), g(t)$.

We can obtain it geometrically via projection from a point $(0,1)$: take a point $(x,y) \in C \setminus \{(0,1)\}$ draw the line passing through $(0,1)$ and (x,y) and see where it intersects the x-axis:

Line between $(0,1)$ and (x,y):

$$aX + bY + c = 0$$

$$\begin{cases} b + c = 0 \\ a\cdot x + b\cdot y + c = 0 \end{cases}$$

$$\to \begin{cases} c = -b \\ a\cdot x + b\cdot y - b = 0 \end{cases} \to \begin{cases} c = -b \\ a\cdot x + b\cdot(y-1) = 0 \end{cases} \to$$

$$\to \begin{cases} c = -b \\ b = -\dfrac{a\cdot x}{y-1} \end{cases} \quad \begin{aligned} &= a\cdot \frac{x}{y-1} \\ &= a\cdot \frac{x}{1-y} \end{aligned}$$

Hence the line is $\left\{ a \cdot X + a \cdot \frac{x}{1-y} \cdot Y + a \cdot \frac{x}{y-1} = 0 \right\}$

i.e. $\left\{ X + \frac{x}{1-y} \cdot Y + \frac{x}{y-1} = 0 \right\}$

Intersection of the line with X-axis : $\begin{cases} X + \frac{x}{1-y} Y + \frac{x}{y-1} = 0 \\ Y = 0 \end{cases}$

$\begin{cases} X = \frac{x}{1-y} \\ Y = 0 \end{cases}$. Hence, the projection gives a map :

$$\pi : C \setminus \{(0,1)\} \longrightarrow \mathbb{C}$$
$$(x,y) \longmapsto \frac{x}{1-y}$$

We claim that this map is actually invertible : we can also construct the inverse geometrically.

- Take $t \in \mathbb{C}$
- Take the line between : $(t,0)$ and $(0,1)$

  parametric representation
  $\{(\lambda \cdot t, 1-\lambda) \mid \lambda \in \mathbb{C}\}$
- Look at the intersection points between the line and C : $\begin{cases} X = \lambda \cdot t \\ Y = 1-\lambda \\ X^2+Y^2 = 1 \end{cases}$

$\rightarrow \begin{cases} X = \lambda t \\ Y = 1-\lambda \\ \lambda^2 t^2 + (1-\lambda)^2 = 1 \end{cases} \rightarrow \begin{cases} X = \lambda t \\ Y = 1-\lambda \\ \lambda^2 t^2 + \cancel{1} + \lambda^2 - 2\lambda = \cancel{1} \end{cases}$

$\rightarrow \begin{cases} X = (1-\lambda)t, \; Y = \lambda \\ \lambda^2(t^2+1) - 2\lambda = 0 \end{cases}$

This line meets the curve C in two points:

- $(0,1)$: corresponding to $\lambda = 0$
- another point, corresponding to $\lambda = \frac{2}{t^2+1}$:
  we can write this point explicitly as

$$x = \frac{2t}{t^2+1} \quad , y = 1 - \frac{2}{t^2+1} = \frac{t^2-1}{t^2+1}$$

This gives us our inverse map

$$\pi : C \setminus \{(0,1)\} \to \mathbb{C} \qquad f : \mathbb{C} \longrightarrow C \setminus \{(0,1)\}$$

$$(x,y) \longmapsto \frac{x}{1-y} \qquad t \longmapsto \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right)$$

We know already from the geometry that these are inverse to each other. But we can also check it explicitly

- $(\pi \circ f)(t) = \pi\left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right) = \frac{2t}{t^2+1} \Big/ 1 - \frac{t^2-1}{t^2+1}$

$$= \frac{2t}{t^2+1-(t^2-1)} = \frac{2t}{2} = t$$

- $(f \circ \pi)(x,y) = f\left(\frac{x}{1-y} \right)$. We should check that

$$x = \frac{2\left(\frac{x}{1-y}\right)}{\left(\frac{x}{1-y}\right)^2+1} = \frac{2x(1-y)}{x^2+(1-y)^2} = \frac{2x(1-y)}{x^2+1+y^2-2y} \overset{[x^2+y^2=1]}{=} \frac{2x\cancel{(1-y)}}{2\cancel{-2y}} = x$$

$$y = \frac{\left(\frac{x}{1-y}\right)^2-1}{\left(\frac{x}{1-y}\right)^2+1} = \frac{x^2-(1-y)^2}{x^2+(1+y)^2} = \frac{x^2-1-y^2+2y}{x^2+1+y^2-2y} \overset{[x^2=1-y^2]}{=} \frac{-2y^2+2y}{2-2y} = y$$

Now let's go back to our integral:

$$\int \frac{1}{y}\,dx \longrightarrow \left[\begin{matrix} x = \dfrac{2t}{t^2+1} \\[2ex] y = \dfrac{t^2-1}{t^2+1} \end{matrix}\right] \longrightarrow \int \frac{t^2+1}{t^2-1}\, d\left(\frac{2t}{t^2+1}\right) dt$$

$$= \int \frac{\cancel{t^2+1}}{t^2-1}\cdot\frac{2(t^2+1)-2t\cdot 2t}{(t^2+1)^{\cancel{2}}}\,dt = \int \frac{-2t^2+2}{t^2-1}\,dt = -2\int dt = -2t$$

The precise formula here at the end does not really matter, because we know that once we express the integral in terms of rational functions, we are done.

[**Remark**: The projection gives also a geometric argument for the parametric formulas $\cos\theta = \dfrac{2t}{t^2+1}$, $\sin\theta = \dfrac{t^2+1}{t^2-1}$.]

In general, the same argument works for all integrals of the form

$$\int R(x, \sqrt{x^2+ax+b})\,dx$$

where R is a rational function.

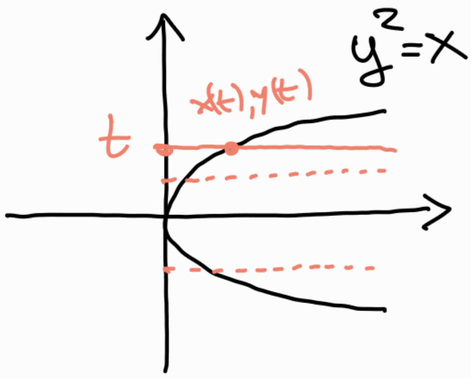
Indeed, in this case, the curve

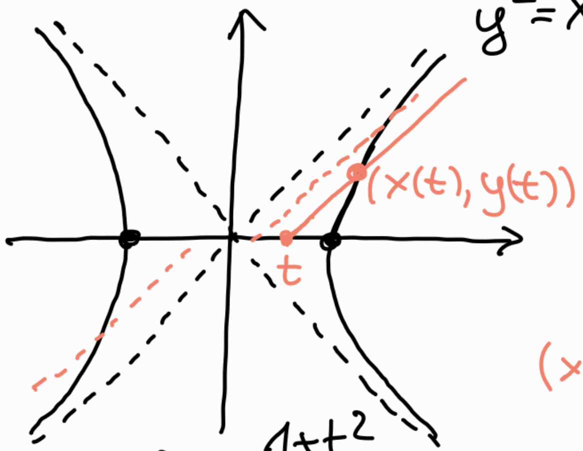
$$C = \{\, y^2 = x^2 + a\cdot x + b \,\}$$

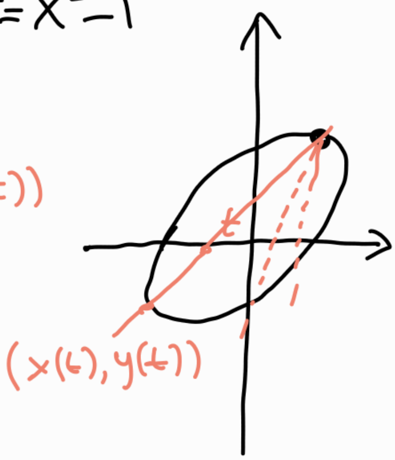
is a RATIONAL CURVE, meaning that it has a parametrization by rational functions

$$x = x(t) \quad , \quad y = y(t)$$

All these parametrizations are also obtained by linear projections







$$x(t) = t^2 \qquad y(t) = t$$

$$x(t) = \frac{1+t^2}{2t} \qquad y(t) = \frac{1-t^2}{2t}$$

What about the integral

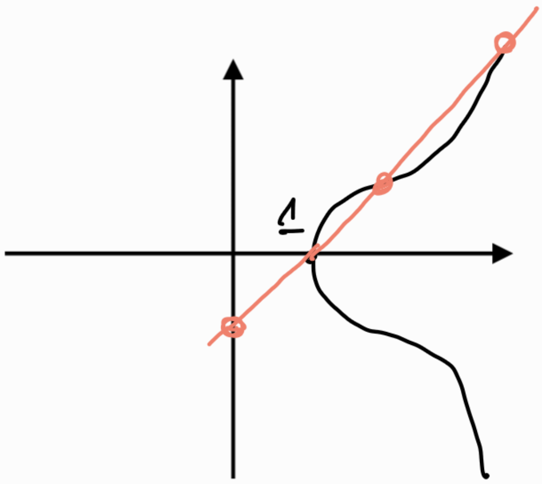
$$\int \frac{1}{\sqrt{x^3-1}}\, dx \quad ?$$

Can we do the same?

To apply the same idea, we would need to find a rational parametrization of the curve

$$C = \{ y^2 = x^3 - 1 \}$$

The real part of this curve looks like this



Let's look for example at the projection from the point $(1,0)$ onto the y-axis

$$\pi : C \setminus \{(1,0)\} \longrightarrow \mathbb{C}$$

$$(x,y) \longmapsto \frac{y}{x-1}$$

Can we reverse this? The picture tells us already that we can't. Let's do this formally:

- Take a point $(0,t)$ on the y-axis
- Take the line between $(0,t)$ and $(+1,0)$: $\{(\lambda+1, \lambda t) \mid \lambda \in \mathbb{C}\}$
- Intersect it with the curve C :

$$\begin{cases} x = \lambda+1 \\ y = \lambda t \\ y^2 = x^3 - 1 \end{cases} \to \begin{cases} x = \lambda+1 \\ y = \lambda t \\ \lambda^2 t^2 = (\lambda+1)^3 - 1 \end{cases} \to \begin{cases} x = \lambda-1 \\ y = \lambda t \\ \lambda^2 t^2 = \lambda^3 + 3\lambda^2 + 3\lambda + 1 - 1 \end{cases}$$

$$\to \begin{cases} x = \lambda - 1 \\ y = \lambda t \\ \lambda^3 - (3+t^2)\lambda^2 + 3\lambda = 0 \end{cases}$$

These are three intersection points!

$$\begin{cases} \lambda = 0 \\ \lambda^2 - (3+t^2)\lambda + 3 = 0 \end{cases}$$

So, the projection π is not injective.

Maybe we can take the projection from another point? Actually, it turns out that the curve C is <u>not</u> rational, so it is impossible to give it a parametrization with rational functions.

<u>Prop</u>: $C = \{ y^2 = x^3 - 1 \}$ is not rational.

<u>proof</u>: suppose it is, then we can find parametrizations

$$x(t) = \frac{A(t)}{B(t)}, \quad y(t) = \frac{C(t)}{D(t)}$$

$A, B, C, D \in \mathbb{C}[t]$ polynomials. Assume that

A, B are coprime and that C, D are coprime. We write

$$\frac{C(t)^2}{D(t)^2} = \frac{A(t)^3}{B(t)^3} - 1 \iff \underset{(*)}{B(t)^3 C(t)^2 = A(t)^3 D(t)^2 - B(t)^3 D(t)^2}$$

We see in particular that $D(t)^2 \mid B(t)^3 C(t)^2$ and since D, C are coprime it must be that $D(t)^2 \mid B(t)^3$.
On the other hand, we also see that $B(t)^3 \mid A(t)^3 D(t)^2$ and since B, A are coprime it must be that $B(t)^3 \mid D(t)^2$
Hence, it must be that $B(t)^3 = \lambda \cdot D(t)^2$ for a certain $\lambda \in \mathbb{C}, \lambda \neq 0$ and up to taking a square root of λ and renaming $D(t)$, we can assume that $B(t)^3 = D(t)^2$. Up to rescaling $C(t)$ and $A(t)$ we can suppose that both $B(t), D(t)$ are monic. Hence it must be that $B(t) = E(t)^2$, $D(t) = E(t)^3$ for a certain $E(t) \in \mathbb{C}[t]$. So we write

$$\frac{C(t)^2}{E(t)^6} = \frac{A(t)^3}{E(t)^6} - 1 \iff C(t)^2 = A(t)^3 - E(t)^6$$

Now we can differentiate and we get

$$2C(t)\dot{C}(t) = 3A(t)^2\dot{A}(t) - 6E(t)^5\dot{E}(t) \quad (**)$$

If we multiply by $C(t)$ we get

$$2C^2\dot{C} = 3A^2C\dot{A} - 6E^5C\dot{E}$$
$$2(A^3 - E^6)\dot{C} = 3A^2C\dot{A} - 6E^5C\dot{E}$$
$$2A^3\dot{C} - 3A^2C\dot{A} = 2E^6\dot{C} - 6E^5C\dot{E}$$

$$A^2(2A\dot C - 3C\dot A) = E^5(2E\dot C - 6C\dot E)$$

Hence $A^2 \mid 2E\dot C - 6C\dot E$ and $E^5 \mid 2A\dot C - 3C\dot A$

Can it be that $2E\dot C - 6C\dot E = 0$? Well, since C, E are coprime, if this happens, then $C \mid \dot C$ and $E \mid \dot E$ but then, by degree reasons, $\dot E = \dot C = 0$ so that E, C are constants, and then A is also constant.

Thus, suppose that they are not constant, then degree reasons give that

$$2 \deg A \leq \deg E + \deg C - 1$$
$$5 \deg E \leq \deg A + \deg C - 1$$

If we multiply (**) by $A(t)$ instead we get

$$2AC\dot C = 3A^3\dot A - 6AE^5\dot E$$
$$2AC\dot C = 3(C^2+E^6)\dot A - 6AE^5\dot E$$
$$2AC\dot C = 3C^2\dot A + 3E^6\dot A - 6AE^5\dot E$$
$$C(2A\dot C - 3C\dot A) = 3E^5(3E\dot A - 6A\dot E)$$

So, as before, we get

$$\deg C \leq \deg E + \deg A - 1$$

So the two previous inequalities give

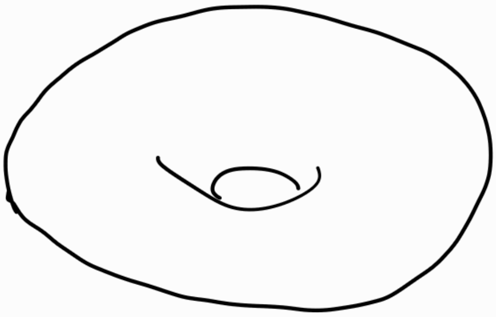
$$\begin{cases} 2\deg A \leq 2\deg E + \deg A - 2 \\ 5\deg E \leq 2\deg A + \deg E - 2 \end{cases}$$

$$\begin{cases} \deg A \leq 2\deg E - 2 \\ 4\deg E \leq 2\deg A - 2 \leq 4\deg E - 4 \end{cases}$$

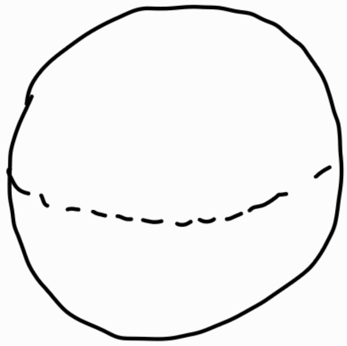
so we arrived at a contradiction. $\square$

This was quite painful. We will see later how to make this proof much easier via geometric methods.

Intuitively: the curve $y^2 = x^3 - 1$ is a Riemann surface of genus 1 while any rational curve has genus 0



$y^2 = x^3 - 1$



rational curve

In general, the integrals like

$$\int \frac{1}{\sqrt{x^3-1}}\,dx$$

are called ELLIPTIC INTEGRALS or, more generally ABELIAN INTEGRALS and a big part of the theory of curves arose from them.