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Fachbereich Mathematik
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Algebraic Curves
WS 2022/23

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Exercise Sheet 9

Please upload your solutions to URM or send them by email by Thursday, January 19 at 10:00

Exercise 9.1 (5 points) Let S be a compact and connected Riemann surface, D a divisor such that $H^0(X,D)$ is base-point-free of dimension $r+1$ and let $|D|$ the corresponding complete linear system. We choose a basis of $H^0(X,D)$ and we define the map

$$\varphi\colon S \to \mathbb{P}^r, \mapsto \varphi = [f_0,\dots,f_r]$$

we want to know when this map is an embedding.

a) Show that the map is injective if and only if $|D|$ separates points. This means that for any two distinct points $p,q \in S$ there is a divisor $E \in |D|$ such that $p \in E$ and $q \notin E$.

b) Show that the differential of φ is everywhere injective if and only if $|D|$ separates tangent vectors. This means that for every point $p \in S$ there is a divisor $E \in |D|$ such that $\mathrm{ord}_p(D) = 1$. Express this condition also in terms of the functions in $H^0(S,D)$.

c) Prove that φ is an embedding, that is is injective with injective differential, if and only if

$$h^0(S, D-p-q) = h^0(S,D) - 2$$

for any two points $p,q \in S$, possibly coincident.

Exercise 9.2 (5 points) Let $E = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ be a complex torus and let $p = \left[\frac{1}{2}+\frac{1}{2}\tau\right]$ be the zero of the theta function $\theta(z,\tau)$. Let also $n \geq 1$ be a positive integer.

a) Prove that $h^0(E, n\cdot p) = n$. Conclude that if q is any point on E, then $h^0(E, n\cdot q) = n$. [*Hint:* Exercise 8.3 and translations].

b) Let D be a divisor on E of positive degree n. Prove that $h^0(E,D) = n$.