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Fachbereich Mathematik
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Algebraic Curves
WS 2022/23

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Exercise Sheet 8

Please upload your solutions to URM or send them by email by Wednesday, December 21 at 14:00

Exercise 8.1 (3 points) Let S be a compact and connected Riemann surface and let $p \in S$ be a point. Consider nonzero meromorphic functions $f_1, \dots, f_n$ on S such that their orders at p are all distinct, i.e. $\operatorname{ord}_p(f_i) \neq \operatorname{ord}_p(f_j)$ if $i \neq j$. Prove that the f_i are linearly independent over $\mathbb{C}$.

Exercise 8.2 (3 points) Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree $2g+2$ with distinct roots. Let X be the hyperelliptic curve given by $\{y^2 = f(x)\}$. Recall that X is defined by gluing together the two charts $\{y^2 = f(x)\}$ and $\{v^2 = g(u)\}$ where $g(u) = u^{2g+2} \cdot f\left(\frac{1}{u}\right)$ and the gluing is given by $u = \frac{1}{x}, v = \frac{y}{x^{g+1}}$.

a) For any $a \in \mathbb{C}$, compute the divisor of the rational function $x - a$ on X.

b) Compute the divisor of the function y.

Exercise 8.3 (4 points) Let $\tau \in \mathcal{H}$ be a complex number with positive imaginary part and let $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be the corresponding complex torus. We also denote by $\theta(z) = \theta(z, \tau)$ the corresponding theta function, and by $p = \left[\frac{1}{2} + \tau\frac{1}{2}\right]$ its unique zero on E.

a) Consider the higher logarithmic derivatives $\frac{d^n \log\theta}{dz^n}$ for $n \geq 2$. Prove that they define meromorphic functions on E.

b) Show moreover that $\frac{d^n \log\theta}{dz^n}$ has a pole of order n at p and no other poles.