Exercise Sheet 6

Please upload your solutions to URM or send them by email by Wednesday, December 7 at 14:00

Exercise 6.1 (4 points)

a) Let $f\colon S_1 \to S_2$ be a nonconstant holomorphic map of degree 1 between compact and connected Riemann surfaces. Prove that f is an isomorphism, meaning that it is invertible and that the inverse is holomorphic.

b) Let S_1, S_2 be two compact connected Riemann surfaces of genus g_1, g_2 respectively. Assume that $g_1 < g_2$ and show that every holomorphic map $f\colon S_1 \to S_2$ must be constant.

Exercise 6.2 (6 points) Here we want to prove Fermat's last theorem for polynomials. More precisely, take an integer $d \geq 3$ and let $P(t), Q(t), R(t) \in \mathbb{C}[t]$ be polynomials such that

$$P(t)^d + Q(t)^d = R(t)^d$$

we want to show that the polynomials must be constant.

a) Reduce to the case where $P(t), Q(t), R(t)$ have no common factor, so that we can define a map $f\colon \mathbb{C} \to \mathbb{P}^2$, $f(t) = [P(t), Q(t), R(t)]$. Show that this is holomorphic.

b) Show that the map extends to $\mathbb{P}^1$, meaning that there exists a holomorphic map $F\colon \mathbb{P}^1 \to \mathbb{P}^2$ such that $F_{|\mathbb{C}} = f$.

c) Conclude that the polynomials must be constant [*Hint:* you can use something from Exercise 6.1].