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Fachbereich Mathematik
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Algebraic Curves
WS 2022/23

Exercise Sheet 5

Please upload your solutions on URM or send them by email by Wednesday, November 30 at 14:00

Exercise 5.1 (3 points) Let S be a compact and connected Riemann surface. Prove that any holomorphic map $f\colon S \to \mathbb{C}$ is constant.

Exercise 5.2 (3 points) Let $C = \{f(x,y) = 0\}$ be a smooth plane affine curve passing through $O = (0,0)$, which we can consider as a Riemann surface. Consider the projection onto the horizontal axis $\pi\colon C \to \mathbb{C}, \pi(x,y) = x$, and assume that C is not the vertical axis, so that the projection is not constant. Prove that

$$\mathrm{mult}_O(\pi) = \mu_O(C, L)$$

where on the left we take the multiplicity of a holomorphic map between Riemann surfaces and on the right we take the algebraic intersection multiplicity between the curve C and the vertical axis $L = \{x = 0\}$.

Exercise 4.3 (4 points) Let $C = \{F(X,Y,Z) = 0\}$ be a smooth plane projective curve of degree d. A point $p \in C$ is called a *flex point* if C and the tangent line $L = \mathbb{T}_p C$ intersect with multiplicity at least three at p, meaning that $\mu_p(C, \mathbb{T}_p C) \geq 3$. On the other hand, the *Hessian* matrix HF of F is the symmetric matrix whose entries are the second partial derivatives of F.

a) Prove that a point $p \in C$ is a flex point if and only if the determinant of HF vanishes at p [*Hint:* choose coordinates that make your life easier]

b) Assume that $d \geq 2$, so that C is not a line. Prove that C has $3d(d-2)$ flex points, counted with an appropriate notion of multiplicity.