Exercise Sheet 4

Please upload your solutions on URM or send them by email by Wednesday, November 23 at 14:00

Exercise 4.0 (Will be discussed in class) We want to finish the proof from last week. Let $F(x,y), G(x,y)$ two polynomials with multiplicity at 0 given by m, n respectively. Then we can write them as sums of their homogeneous terms $F = F_m + F_{m+1} + \dots$ and $G = G_n + G_{n+1} + \dots$. Assume that F_m, G_n are coprime and let $\mathfrak{m} = (x,y)$. We want to prove that $\mathfrak{m}^{n+m} \subseteq (F,G)$ in $\mathbb{C}[x,y]_{\mathfrak{m}}$.

a) Show that for any $d \geq m+n$ we have an exact sequence of vector spaces

$$0 \to \mathbb{C}[x,y]_{d-m-n} \to \mathbb{C}[x,y]_{d-n} \oplus \mathbb{C}[x,y]_{d-m} \to \mathbb{C}[x,y]_d \to \mathbb{C}[x,y]_d/(F_m,G_n)_d \to 0$$

and conclude that $\mathbb{C}[x,y]_d = (F_m,G_n)_d$.

b) Show that $\mathfrak{m}^{m+n} \subseteq (F,G) + \mathfrak{m}^{m+n+1}$ in $\mathbb{C}[x,y]$.

c) Conclude that $\mathfrak{m}^{n+m} \subseteq (F,G)$ in $\mathbb{C}[x,y]_{\mathfrak{m}}$.

Exercise 4.2 (10 points divided in three steps)

Here we wanto to study the so called *blow-up* of $\mathbb{C}^2$ at the origin $O = (0,0)$. This is

$$X = \{((z,w),[x_0,x_1]) \in \mathbb{C}^2 \times \mathbb{P}^1(\mathbb{C}) \mid zx_1 = wx_0\}.$$

a) (4 points) Let us show that X is a complex manifold of dimension two. Clearly X is a closed set of $\mathbb{C}^2 \times \mathbb{P}^1(\mathbb{C})$. Put on X the induced topology. One can cover X with two open sets $X = U_0 \cup U_1$ where

$$U_0 = \{((z,w),[x_0,x_1]) \in X \mid x_0 \neq 0\}, \quad U_1 = \{((z,w),[x_0,x_1]) \in X \mid x_1 \neq 0\}$$

We also have maps $\varphi_0 \colon U_0 \to \mathbb{C}^2, \varphi_1 \colon U_1 \to \mathbb{C}^2$ defined by

$$\varphi_0((z,w),[x_0,x_1]) = (z, \frac{x_1}{x_0}), \qquad \varphi_1((z,w),[x_0,x_1]) = (w, \frac{x_0}{x_1})$$

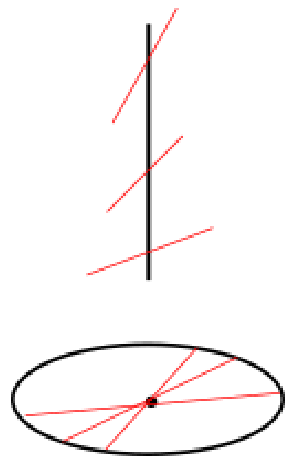
- Prove that U_0, U_1 are charts of X, so that X is a complex manifold of dimension two.
- Prove that the projection map $\pi \colon X \to \mathbb{C}^2$ onto the (z,w) coordinates is holomorphic.
- Prove that $E \colon = \pi^{-1}(O) \cong \mathbb{P}^2$ and that $\pi \colon X \setminus E \to \mathbb{C}^2 \setminus \{O\}$ is an isomorphism.

b) (2 points) Let $L \subseteq \mathbb{C}^2$ be a line passing through O with equation $L = \{az + bw = 0\}$ and consider its *strict transform* $\widetilde{L} \subseteq X$ defined as:

$$\widetilde{L} = \overline{\pi^{-1}(L \setminus \{O\})}.$$

- Show that $\widetilde{L} = \{((z,w),[b,-a]) \mid az + bw = 0\}$.
- Show that $\pi^{-1}(L) = \widetilde{L} \cup E$ and describe the set of points of $\widetilde{L} \cap E$.

From your solution it will follow that the map $L \mapsto \widetilde{L} \cap E$ gives a bijection between lines through the origin of $\mathbb{C}^2$ and points in E.



c) (4 points) Consider the plane curve $C = \{w^2 = z^3 + z^2\}$ and let $\widetilde{C} \subseteq X$ be its strict transform that is defined as $\widetilde{C} = \overline{\pi^{-1}(C \setminus \{O\})}$

- Show that $\widetilde{C} \cap E$ consists of two points, corresponding to the tangent lines to C at O.
- Show that $\widetilde{C}$ is naturally a Riemann surface and that it is isomorphic to $\mathbb{C}$.