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Fachbereich Mathematik
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Algebraic Curves
WS 2022/23

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Exercise Sheet 2

Please upload your solutions on URM or send them by email by Thursday, November 10 at 14:00

Exercise 2.1 (4 points) For the following pairs of plane projective curves C, D, compute their decomposition into irreducible components, compute the intersection points $C \cap D$ and the intersection multiplicity at each point.

a) $C = \{X^2 + Y^2 + 2YZ = 0\}$ and $D = \{Y^3X^6 - Y^6X^2Z = 0\}$.

b) $C = \{XZ + Y^2 = 0\}$ and $D = \{XZ^2 + Y^2Z - X^3 = 0\}$.

c) $C = \{Y^2 - X^2 + Z^2 = 0\}$ and $D = \{Y^2 - X^2 + 2YZ + Z^2 = 0\}$.

Exercise 2.2 (3 points) Consider two affine real plane curves $C(\mathbb{R}) = \{F(x, y) = 0\}$ and $D(\mathbb{R}) = \{G(x, y) = 0\}$ where $F(x, y), G(x, y) \in \mathbb{R}[x, y]$ are real polynomials. We define their real intersection multiplicity at a point $P \in \mathbb{A}^2(\mathbb{R})$ as $\mu_P(C, D) = \dim_{\mathbb{R}} \mathcal{O}_P/(F, G)$. We can then define the same notion for projective real plane curves by reducing to the affine case, as for complex curves. Using the complex Bezout's theorem prove that if $C(\mathbb{R})$ and $D(\mathbb{R})$ are two real projective plane curves, then

$$\sum_{P \in C(\mathbb{R}) \cap D(\mathbb{R})} \mu_P(C, D) \equiv \deg C(\mathbb{R}) \cdot \deg D(\mathbb{R}) \mod 2$$

In particular, two real projective curves of odd degree always have an intersection point.

Exercise 2.3 (3 points) Let $C = \{F(X, Y, Z) = 0\}$ be a reduced and irreducible complex projective plane curve of degree $d > 1$. A line in $\mathbb{P}^2$ corresponds to a linear form $\ell(X, Y, Z) = aX + bY + cZ$ up to multiplication by a nonzero scalar, hence these lines are parametrized themselves by the so-called dual projective plane $(\mathbb{P}^2)^*$ with coordinates $[a, b, c]$. Bezout's theorem tells us that a line $L \in \mathbb{P}^2$ intersects C in d points, counted with multiplicity. Prove that the set $Z \subseteq (\mathbb{P}^2)^*$ of lines which intersect C in less than d distinct points (without counting the multiplicity) is Zariski closed. [*Hint*: look up the discriminant of a univariate polynomial].