Exercise 1.1

(a) Write

$$F(X,Y,Z) = aX^2 + 2bXY + 2cXZ + dY^2 + 2eYZ + fZ^2$$

for $a,b,c,d,e,f \in \mathbb{C}$. Then

$$F(X,Y,Z) = (X\ Y\ Z)\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

(b) Recall from linear algebra that a symmetric matrix $A \in \mathbb{C}^{3\times 3}$ can be always brought into diagonal form by a change of basis. More precisely, there is $S \in GL(3,\mathbb{C})$ such that SAS^t is diagonal with only 1 or 0 on the diagonal. This matrix S induces a linear of coordinates on $\mathbb{P}^2$, which does not change the fact that a conic is a double line, a sum of lines, or reduced and irreducible.

Hence, we can suppose that A is diagonal with only 1 and 0 on the diagonal.

Now it is easy to conclude the proof:

- rank 1: $A = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$ so

$F(X,Y,Z) = X^2$ defines a double line

$$Q = \{X^2 = 0\} = 2\{X=0\} = 2L$$

- rank 2: $A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$ so

$F(X,Y,Z) = X^2+Y^2 = (X+iY)\cdot(X-iY)$

defines the sum of two distinct lines

$$Q = \{X^2+Y^2=0\} = \{X+iY=0\} + \{X-iY=0\}$$
$$= L_1 + L_2$$

- rank 3: $A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ so

$F(X,Y,Z) = X^2+Y^2+Z^2$ is an irreducible polynomial, for example because of Eisenstein's criterion, so the curve

$Q = \{F=0\}$ is reduced and irreducible.

Exercise 1.2 :

(a) We prove that $\mathbb{P}^n(k)$ is compact: consider the projection map

$\pi: k^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n(k)$, $v \longmapsto [v]$

and restrict it to the sphere

$S^{n+1} = \{ v \in k^{n+1} \mid \|v\| = 1 \}$

where $\|\cdot\|$ is the norm inducing the Euclidean topology on k^{n+1}. We claim that the map

$\pi : S^{n+1} \to \mathbb{P}^n$ is surjective : indeed

let $[v] \in \mathbb{P}^n$ be represented by $v \in k^{n+1}$, $v \neq 0$.

Then $[v] = \left[\frac{v}{\|v\|} \right]$ and $\frac{v}{\|v\|} \in S^{n+1}$.

Since S^{n+1} is compact, it follows that $\mathbb{P}^n$ is compact as well.

Let us prove that $\mathbb{P}^n$ is Hausdorff : first we observe that the Euclidean topology is finer than the Zariski topology. This means that if $\mathbb{V}(F_1, .., F_s) \subseteq \mathbb{P}^n$ is a Zariski closed subset, then it is also an Euclidean closed subset. This is easy to prove :

$\pi^{-1}(\mathbb{V}(F_1, .., F_s)) = \{ v \in k^{n+1} \setminus \{0\} \mid F_1(v) = .. = F_s(v) = 0 \}$

is closed w.r.t. the Euclidean topology on

$k^{n+1}\setminus\{0\}$. Now we observe that if we take an hyperplane $H\subseteq\mathbb{P}^n$ given by an equation $H=\{\ell(x_0,\dots,x_n)=0\}$, then we have two maps

$$\begin{aligned}\mathbb{P}^n\setminus H &\longrightarrow \{(y_0,\dots,y_n)\in\mathbb{A}^{n+1} \mid \ell(y_0,\dots,y_n)=1\}\\ [x_0,\dots,x_n] &\longmapsto \left(\frac{x_0}{\ell(x_0,\dots,x_n)},\dots,\frac{x_n}{\ell(x_0,\dots,x_n)}\right)\end{aligned}$$

$$\begin{aligned}\{(y_0,\dots,y_n)\in\mathbb{A}^{n+1} \mid \ell(y_0,\dots,y_n)=1\} &\longrightarrow \mathbb{P}^n\setminus H\\ (y_0,\dots,y_n) &\longmapsto [y_0,\dots,y_n]\end{aligned}$$

which are continuous in the Zariski and Euclidean topology (since are defined by rational functions which are continuous also in the Euclidean topology) and inverse to one another. Hence these are homeomorphisms

$$\mathbb{P}^n\setminus H \cong \{(y_0,\dots,y_n)\in\mathbb{A}^{n+1} \mid \ell(y_0,\dots,y_n)=1\}\cong k^n$$

Now take two distinct points $p,q\in\mathbb{P}^n$: if we can find a hyperplane that does not contain any of them, we are done, because then p,q are contained in the open subset $\mathbb{P}^n\setminus H$ which is isomorphic to k^n, hence Hausdorff. We are left to prove this useful

Lemma: Let k be an infinite field and let $p_1, \dots, p_m \in \mathbb{P}^n(k)$ be distinct points. Then there is a hyperplane that does not contain any of them.

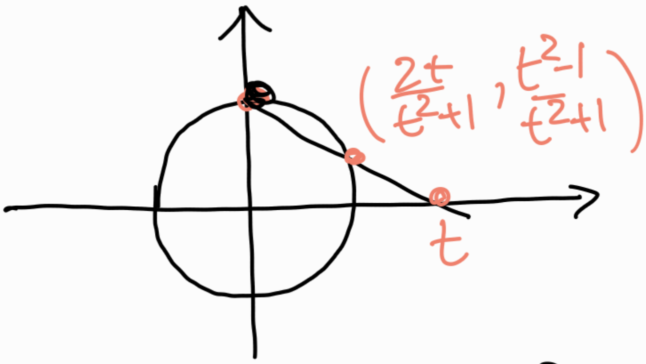
Proof: choose representatives $v_1, \dots, v_m \in k^{n+1}$ for $p_1, \dots, p_m$. Then we need to prove that there is a linear form $\ell(x_0, \dots, x_n) = a_0 x_0 + \dots + a_n x_n$ s.t. $\ell(v_i) \neq 0$ $\forall i = 1, \dots, m$. Let $W = k[x_0, \dots, x_n]_1$ be the space of linear forms: this is a vector space and we can define the vector subspaces $W_i = \{ \ell(x_0, \dots, x_n) \mid \ell(v_i) = 0 \}$.
We need to prove that $W \neq \bigcup_{i=1}^{n} W_i$.
However, no vector space over infinite field is the union of finitely many vector subspaces.

(b) The proof above what somewhat long, but we can use it to solve this point as well: Indeed, consider in $\mathbb{P}^1$ the point $[0,1]$: this is the hyperplane $\{[0,1]\} = \{x_0 = 0\}$. So we have seen above that $\mathbb{P}^1 \setminus \{[0,1]\}$ is homeomorphic to k in the Euclidean topology. Thus since $\mathbb{P}^1$ is compact and

Hausdorff, it must be the Alexandrov compactification of k. This means that if we find another space X which is compact and Hausdorff and that it has a point p s.t. $X \setminus \{p\}$ is homeomorphic to k, then X is homeomorphic to $\mathbb{P}^1(k)$.

- $\underline{k = \mathbb{R}}$: $X = S^1$, $\mathbb{R} \longrightarrow S^1 \setminus \{(0,1)\}$

$$t \longmapsto \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right)$$

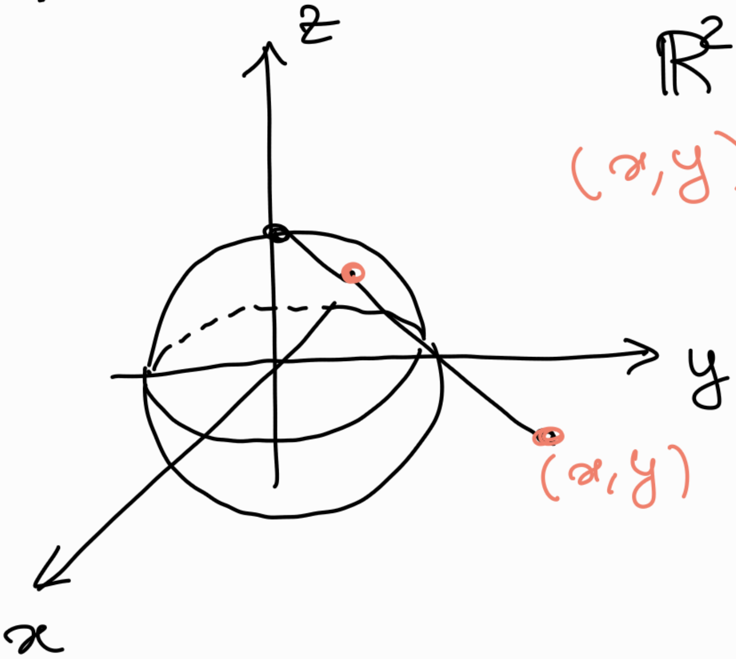


is a homeomorphism.

- $\underline{k = \mathbb{C}}$: $X = S^2$, $\mathbb{C} \to \mathbb{R}^2$, $z \longmapsto (\mathrm{Re}\, z, \mathrm{Im}\, z)$ is a homeomorphism

$$\mathbb{R}^2 \longrightarrow S^2 \setminus \{(0,0,1)\}$$

$$(x,y) \longmapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right)$$



is a homeomorphism.

Exercise 1.3

(a) We have to prove that for any homogeneous F

$$\deg(F)\cdot F = X\cdot\frac{\partial F}{\partial X} + Y\cdot\frac{\partial F}{\partial Y} + Z\cdot\frac{\partial F}{\partial Z}$$

Observe that if this holds it for two homogeneous polynomials F, G of the same degree d, then it holds also for the sum $F+G$. Indeed

$$\deg(F+G)\cdot(F+G) = d\cdot(F+G)$$
$$= dF + dG = X\cdot\frac{\partial F}{\partial X} + Y\cdot\frac{\partial F}{\partial Y} + Z\cdot\frac{\partial F}{\partial Z}$$
$$+ X\cdot\frac{\partial G}{\partial X} + Y\cdot\frac{\partial G}{\partial Y} + Z\cdot\frac{\partial G}{\partial Z}$$
$$= X\cdot\frac{\partial(F+G)}{\partial X} + Y\cdot\frac{\partial(F+G)}{\partial Y} + Z\,\frac{\partial(F+G)}{\partial Z}.$$

Moreover, if it holds for two polynomials of possibly different degrees F, G, then it also holds for their product $F\cdot G$. Indeed:

$$X\cdot\frac{\partial(F\cdot G)}{\partial X} + Y\,\frac{\partial(F\cdot G)}{\partial Y} + Z\cdot\frac{\partial(F\cdot G)}{\partial Z} =$$
$$= X\left(\frac{\partial F}{\partial X}\cdot G + F\cdot\frac{\partial G}{\partial X}\right) + Y\left(\frac{\partial F}{\partial Y}\cdot G + F\cdot\frac{\partial G}{\partial Y}\right)$$
$$+ Z\cdot\left(\frac{\partial F}{\partial Z}\cdot G + F\cdot\frac{\partial G}{\partial Z}\right) =$$

$$= G\cdot\left(X\frac{\partial F}{\partial X}+Y\frac{\partial F}{\partial Y}+Z\frac{\partial F}{\partial Z}\right)$$
$$+ F\cdot\left(X\cdot\frac{\partial G}{\partial X}+Y\cdot\frac{\partial G}{\partial Y}+Z\cdot\frac{\partial G}{\partial Z}\right)$$

$$= \deg(F)\cdot FG + \deg(G)\cdot FG$$
$$= [\deg(F)+\deg(G)]\cdot FG = \deg(FG)\cdot FG$$

Hence it is enough to prove it for the variables X, Y, Z, where it is obvious.

(b) We apply the Euler identity to the derivatives, and we get

$$(d-1)\frac{\partial F}{\partial X} = X\cdot\frac{\partial^2 F}{\partial X^2}+Y\cdot\frac{\partial^2 F}{\partial X\partial Y}+Z\cdot\frac{\partial^2 F}{\partial X\partial Z}$$
$$(d-1)\frac{\partial F}{\partial Y} = X\cdot\frac{\partial^2 F}{\partial X\partial Y}+Y\cdot\frac{\partial^2 F}{\partial X^2}+Z\cdot\frac{\partial^2 F}{\partial Y\partial Z}$$
$$(d-1)\frac{\partial F}{\partial Z} = X\cdot\frac{\partial^2 F}{\partial X\partial Z}+Y\cdot\frac{\partial^2 F}{\partial Y\partial Z}+Z\,\frac{\partial^2 F}{\partial Z^2}$$

which is exactly what we want to prove.