Exercise Sheet 10

Please upload your solutions to URM or send them by email by Thursday, January 26 at 10:00

Exercise 10.1 (5 points) Let $C = \{F(X,Y,Z) = 0\}$ be a smooth plane projective curve of degree $d \geq 3$. In the affine chart $\{Z \neq 0\}$ we obtain the affine curve $C \cap \{Z \neq 0\} = \{f(x,y) = 0\}$ where $f(x,y) = F(x,y,1)$. Prove that a basis of the space $H^0(C, \omega_C)$ of holomorphic differential forms is given by

$$\omega_{a,b} = \frac{x^a y^b}{f_y} dx, \qquad \text{for } a, b \in \mathbb{N}, a + b \leq d - 3$$

where $f_y = \frac{\partial f}{\partial y}$. In particular, you should show how these forms, a priori defined only on $C \cap \{Z \neq 0\}$, extend to the whole curve C.

Exercise 10.2 (5 points) Take $\tau \in \mathcal{H}$ and the corresponding complex torus $X = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$. We also take the theta function $\theta(z) = \theta(\tau, z)$ and its zero $p = \left[\frac{1}{2} + \frac{1}{2}\tau\right]$. The embedding induced by $D = 3p$ is

$$\varphi \colon X \hookrightarrow \mathbb{P}^2, \qquad \varphi = \left[1, \frac{\partial^2 \log \theta}{\partial z^2}, \frac{\partial^3 \log \theta}{\partial z^3}\right]$$

a) Show that a basis of $H^0(X, 4p)$ is given by $1, \frac{\partial^2 \log \theta}{\partial z^2}, \frac{\partial^3 \log \theta}{\partial z^3}, \left(\frac{\partial^2 \log \theta}{\partial z^2}\right)^2$

b) Show that $\left(\frac{\partial^3 \log \theta}{\partial z^3}\right)^2 + 4\left(\frac{\partial^2 \log \theta}{\partial z^2}\right)^3 \in H^0(X, 4p)$, possibly via computer algebra. Deduce that

$$\left(\frac{\partial^3 \log \theta}{\partial z^3}\right)^2 + 4\left(\frac{\partial^2 \log \theta}{\partial z^2}\right)^3 = a\left(\frac{\partial^2 \log \theta}{\partial z^2}\right)^2 + b \cdot \frac{\partial^2 \log \theta}{\partial z^2} + c + d \cdot \frac{\partial^3 \log \theta}{\partial z^3}$$

for certain $a, b, c, d \in \mathbb{C}$.

c) Looking at the involution $\sigma \colon \mathbb{C} \to \mathbb{C}, \sigma(z) = -z$, deduce that $d = 0$.

d) Conclude that $\varphi \colon X \hookrightarrow \mathbb{P}^2$ gives an isomorphism of X with the elliptic curve $E = \{y^2 = -4x^3 + ax^2 + bx + c\}$.