Exercise Sheet 1

Please upload your solutions on URM by Wednesday, November 2 at 14:00

Exercise 1.1 (4 points) A complex projective plane curve $Q \subseteq \mathbb{P}^2$ of degree two is called a quadric or a conic.

a) Show that any homogeneous polynomial $F(X, Y, Z)$ of degree two can be written in the form

$$F(X,Y,Z) = \begin{pmatrix} X & Y & Z \end{pmatrix} \cdot A \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

where A is a 3×3 symmetric matrix. Hence, plane conics correspond to 3×3 symmetric matrices, up to multiplication by a nonzero scalar.

b) The rank of a plane conic is defined to be the rank of a corresponding symmetric matrix. Show that a conic has rank 1 if and only if it is of the form $Q = 2L$ where L is a line, that it has rank 2 if and only if it is of the form $Q = L_1 + L_2$ where L_1, L_2 are two distinct lines, and that it has rank 3 if and only if it is reduced and irreducible.

Exercise 1.2 (3 points) Let $k = \mathbb{R}$ or $k = \mathbb{C}$. Then the projective space $\mathbb{P}^n(k)$ has also an Euclidean topology, which is simply the quotient topology $\mathbb{P}^n = \left(k^{n+1} \setminus \{0\}\right)/(k)^*$ that makes the projection map $\pi \colon k^{n+1} \setminus \{0\} \to \mathbb{P}^n(k)$ continuous. Here we put the Euclidean topology on k^{n+1}.

a) Prove that the real or complex projective space $\mathbb{P}^n(k)$ is compact and Hausdorff with the Euclidean topology.

b) Prove that $\mathbb{P}^1(\mathbb{R})$ is homeomorphic to the circle S^1 whereas $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to the sphere S^2. [*Hint*: you can look up and use the properties of the *Alexandrov compactification*]

Exercise 1.3 (3 points) Let $F(X, Y, Z) \in \mathbb{C}[X, Y, Z]$ be a homogeneous polynomial of degree d.

a) Let $\nabla F = \left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}\right)$ be the row vector of partial derivatives of F. Prove the Euler identity:

$$d \cdot F = \nabla F \cdot \begin{pmatrix} X & Y & Z \end{pmatrix}^t$$

b) Let HF be the Hessian matrix of the second partial derivatives of F. Prove that

$$(d-1) \cdot \nabla F = \begin{pmatrix} X & Y & Z \end{pmatrix} \cdot HF.$$