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Fachbereich Mathematik
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Algebraic Curves
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Exercise Sheet 0

This sheet will just be discussed in class

Exercise 0.1 We have seen in the lectures that the curve $C = y^2 = x^3 - 1$ is not rational, and in particular the projection from a point $\pi\colon C \setminus \{P\} \to \mathbb{C}$ will be never injective. This is true for almost all the curves defined by a polynomial of degree three, but not for all of them. Some of these are actually rational.

a) Consider the curve $C = \{y^2 = x^3\}$ and try to draw the real part of the curve. Consider the linear projection from the point $O = (0,0) \in C$ to the line $L = \{x = 1\}$ and use it to give a rational parametrization of C.

b) Consider the curve $C = \{y^2 = x^3 + x^2\}$ and try to draw the real part of the curve. Consider the linear projection from the point $O = (0,0) \in C$ to the line $L = \{x = 1\}$ and use it to give a rational parametrization of C.

Exercise 0.2 Show that any rational function $p(x)/q(x) \in \mathbb{C}(x)$ can be written as

$$\frac{p(x)}{q(x)} = r(x) + \sum_{i=1}^{n} \frac{a_i}{(x - b_i)^{m_i}}$$

for $r(x) \in \mathbb{C}[x]$ a polynomial, $a_i, b_i \in \mathbb{C}$ and $m_i \in \mathbb{N}$.

Exercise 0.3 Consider two distinct irreducible subsets $\mathbb{V}(f), \mathbb{V}(g) \subseteq \mathbb{A}^2$. We want to show that the intersection $\mathbb{V}(f,g)$ is finite.

a) Observe that the two polynomials $f(x,y)$ and $g(x,y)$ have no common factors in $\mathbb{C}[x,y]$ so that they also have no common factors in $K[y]$ where $K = \mathbb{C}(x)$ is the field of rational functions in x.

b) Since $K[y]$ is a PID, there are $a(y), b(y) \in K[y]$ such that $a(y)f(x,y) + b(y)g(x,y) = 1$. Use this to conclude.

As a consequence prove

c) The irreducible closed subsets $X \subseteq \mathbb{A}^2$ are either a single point, a curve of the form $\mathbb{V}(f)$, where f is irreducible, and the whole space.