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Algebraic Geometry
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Exercise Sheet 9

Submit by: Monday, 27/06/22, 10 am

Exercise 9.1 Consider the twisted cubic curve $C \subseteq \mathbb{P}^3$ given by

$$C = \left\{ \operatorname{rk} \begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \end{pmatrix} \leq 1 \right\}$$

This is isomorphic to $\mathbb{P}^1$ so it should be smooth. Prove it starting from the equations.

Exercise 9.2 Let $f(X_0, \ldots, X_n)$ be a homogeneous polynomial of degree d. Prove that

$$\sum_{j=0}^{n} \frac{\partial f}{\partial X_j} \cdot X_j = d \cdot f.$$

In particular, if all the partial derivatives vanish at a point, then the polynomial vanishes at the point as well.

Exercise 9.3 Consider the space $V_{n,d} = \mathbb{P}\left(K[X_0, \ldots, X_n]\right) \cong \mathbb{P}^{\binom{n+d}{d}-1}$ that parametrizes all nonzero homogeneous polynomials of degree d in $n+1$ variables up to a scalar. Inside it we have the locus of singular surfaces

$$\Sigma_{n,d} = \left\{ [f] \in V_{n,d} \mid \text{ there exists } p \in \mathbb{P}^n \text{ such that } \frac{\partial f}{\partial X_j}(p) = 0 \text{ for all } j = 0, \ldots, n \right\}$$

and the corresponding locus $Z_{n,d} \subseteq V_{n,d} \times \mathbb{P}^n$ given by

$$Z_{n,d} = \left\{ ([f], p) \in V_{n,d} \times \mathbb{P}^n \mid \frac{\partial f}{\partial X_j}(p) = 0 \text{ for all } j = 0, \ldots, n \right\}$$

a) Consider the second projection $pr_2 \colon Z_{n,d} \to \mathbb{P}^n$ and prove that all the fibers are projective spaces of dimension $\binom{n+d}{d} - n - 2$. This implies that $Z_{n,d}$ is irreducible of dimension $\binom{n+d}{d} - n - 1$. Conclude that $\Sigma_{n,d}$ is irreducible.

b) Give explicit equations for $\Sigma_{n,2}$.

Exercise 9.4 Let X be a variety and $f \in \mathcal{O}(X)$ a nonzero regular function. We consider the corresponding hypersurface $D = \mathbb{V}(f)$ and a point $p \in D$.

a) Show that $\dim T_pD \geq \dim T_pX - 1$.

b) Show that if D is smooth at p then X is smooth at p as well. Show also that the converse does not need to hold.

c) Consider the quadric cone $X = \{z^2 = xy\} \subseteq \mathbb{A}^3$ and the line $L = \{x = z = 0\} \subseteq X$. Show that L cannot be cut out by a single regular function on X.