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Algebraic Geometry
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Exercise Sheet 7

Submit by: Monday, 13/06/22, 10 am

Exercise 7.1 Consider the Veronese embedding

$$v_3 \colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \qquad [s,t] \mapsto [s^3, s^2t, st^2, t^3]$$

the image $C = v_3(\mathbb{P}^1)$ is called the twisted cubic.

a) Let x_0, x_1, x_2, x_3 be the homogeneous coordinates on $\mathbb{P}^3$. Show that

$$C = \left\{ \operatorname{rk} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \leq 1 \right\}$$

b) Let $H \subseteq \mathbb{P}^3$ be a hyperplane, show that $v_3^{-1}(H)$ corresponds to the roots of an homogeneous polynomial of degree three. Conclude that a general hyperplane H intersects C in three points.

c) Let $p_1, p_2, p_3, p_4 \in C$ be distinct points. Show that they are not contained in a hyperplane.

Exercise 7.2 We have seen that we can interpret the second Veronese embedding as

$$v_2 \colon \mathbb{P}(K[x_0,\dots,x_n]_1) \hookrightarrow \mathbb{P}(K[x_0,\dots,x_n]_2), \quad [\ell] \mapsto [\ell^2]$$

a) Give a natural identification of the space $K[x_0,\dots,x_n]_2$ with the space of $(n+1) \times (n+1)$ symmetric matrices with coefficients in K.

b) Under this identification, show that the image of the Veronese embedding corresponds to the set of symmetric matrices of rank exactly one. Give explicit equations for it.

Exercise 7.3 Let $V = K^n$ be the standard vector space over K. A matrix $A \in K^{n\times n}$ is called skew-symmetric if $A^t = -A$. We denote the space of skew-symmetric matrices by $\operatorname{Alt}_n(K)$.

a) Show that any alternating bilinear map $V \times V \to K$ is of the form $(x,y) \mapsto x^t Ay$ for some skew-symmetric matrix A. Give a natural isomorphism $\operatorname{Hom}(\wedge^2 V, K) \cong \operatorname{Alt}_n(K)$ as K-vector spaces.

b) For two functionals $\phi_1, \phi_2 \in V^*$ show that the map

$$\phi_1 \wedge \phi_2 \colon \ \wedge^2 V \to K, \qquad v_1 \wedge v_2 \mapsto \langle \phi_1 \wedge \phi_2, v_1 \wedge v_2 \rangle := \det(\phi_i(v_j))$$

is linear. Prove then that the pairing

$$\wedge^2(V^*) \times \wedge^2 V \to K, \qquad (\phi_1 \wedge \phi_2, v_1 \wedge v_2) \mapsto \langle \phi_1 \wedge \phi_2, v_1 \wedge v_2 \rangle$$

is nondegenerate, so that it induces an isomorphism $\wedge^2(V^*) \cong \operatorname{Hom}(\wedge^2 V, K)$.

c) Points (a) and (b) together show that $\wedge^2(V^*) \cong \mathrm{Alt}_n(K)$, so that we can see the Plücker embedding of $G(2, V^*)$ as going in the projective space of skew-symmetric matrices:

$$G(2, V^*) \hookrightarrow \mathbb{P}(\wedge^2 V^*) \cong \mathbb{P}(\mathrm{Alt}_n(K))$$

Show that the image is given by the skew-symmetric matrices of rank exactly two. You can limit yourself to the case $n = 4$.

Exercise 7.4 Show that $G(2, 4) \subseteq \mathbb{P}^5$ is a quadric hypersurface by giving an explicit equation. You can use a computer if you wish.