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Fachbereich Mathematik
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Algebraic Geometry
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Exercise Sheet 6

Submit by: Monday, 30/05/22, 10 am

Exercise 5.1

a) Let $F\colon X \to \mathbb{A}^m$ be a morphism from an irreducible variety X and let $D \subseteq \mathbb{A}^m$ be an hypersurface. Show that every component of $F^{-1}(D) \subseteq X$ has codimension at most one. Prove the same for a morphism $F\colon X \to \mathbb{P}^m$.

Now consider a morphism $F\colon \mathbb{A}^n \to \mathbb{P}^m$, we want to prove that F has the form $F = [F_0, \ldots, F_m]$ for certain polynomials F_i such that $V(F_1, \ldots, F_m) = \emptyset$. Consider the open affine charts $U_i = \{Y_i \neq 0\}$ in $\mathbb{P}^m$. If $F^{-1}(V_i) = \mathbb{A}^m$ for some i, then we are done (why?). Otherwise, by the previous point, the $V_i := F^{-1}(U_i)$ are complements of hypersurfaces.

b) Show that the induced maps $F_{|V_i}\colon V_i \to U_i$ can be written in the form

$$F_{|V_i}\colon V_i \to U_i \qquad x \mapsto [F_{i0}(x), \ldots, F_{im}(x)]$$

for certain polynomials F_{ij} such that $F_{ii}(x) \neq 0$ on V_i and moreover the polynomials $(F_{i0}, \ldots, F_{im})$ have no common prime factor.

c) Prove that $F_{ii} \cdot F_{jk} = F_{ik} \cdot F_{ji}$. Deduce that $F_{ji} = g_{ji} \cdot F_{ii}$ for a certain polynomial g_{ji} and observe that these polynomials g_{ij} are actually nonzero constant.

d) Observe that

$$F'\colon \mathbb{A}^n \to \mathbb{P}^m \qquad F'(x) = [F_{00}(x), g_{01} \cdot F_{11}(x), \ldots, g_{0m} \cdot F_{mm}(x)]$$

is a morphism and coincides with F on V_0. Conclude that $F = F'$.

Exercise 5.2 We want to show that any morphism $F\colon \mathbb{P}^n \to \mathbb{P}^m$ has the form $F = [F_0, \ldots, F_m]$ where the F_i are homogeneous polynomials of the same degree with no common zero.

a) Consider the standard open subsets $U_i = X_i \neq 0 \subseteq \mathbb{P}^n$. Using the previous exercise, show that $F_{|U_i}$ has the form

$$F_{|U_i} = [F_{i0}, \ldots, F_{im}]$$

where the F_{ih} are homogeneous polynomials of the same degree d_i and with no common factors.

b) Prove that $F_{ih} \cdot F_{jk} = F_{jh} \cdot F_{ik}$. Deduce that $F_{jh} = g_{jh} \cdot F_{ih}$ for some $g_{jh} \in K^*$.

c) Prove that $F_{00}, \ldots, F_{0m}$ have no common zero and conclude that the morphism F is given by $F = [F_{00}, \ldots, F_{0m}]$.

Exercise 5.3 A plane curve of degree d is a subvariety $C \subseteq \mathbb{P}^2$ given by

$$C = \{F(X, Y, Z) = 0\},$$

where $F(X, Y, Z)$ is an homogeneous polynomial of degree d.

a) Show that plane curves of degree d are naturally parametrized by a projective space $\mathbb{P}^{N_d}$ of dimension $N_d = \binom{d+2}{2} - 1$.

b) A plane curve C as above is called singular if there exists a point $p \in \mathbb{P}^2$ such that

$$F(p) = \frac{\partial F}{\partial X}(p) = \frac{\partial F}{\partial Y}(p) = \frac{\partial F}{\partial Z}(p) = 0$$

Show that the locus $S_d \subseteq \mathbb{P}^{N_d}$ of singular plane curves is closed.

c) Can you give explicit equations for the locus S_2 of singular plane quadrics? In general, the locus S_d is actually an hypersurface in $\mathbb{P}^{N_d}$ and its defining equation is called the discriminant.