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Algebraic Geometry
SS 2022

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Exercise Sheet 4

Submit by: Monday, 09/05/22, 10 am

Exercise 4.1 Let $f\colon X \to Y$ be a morphism of affine varieties and $f^*\colon A(Y) \to A(X)$ the corresponding morphism of the coordinate rings.

a) Let $I \subseteq A(Y)$ be an ideal and let $V(I) \subseteq Y$ be the corresponding closed subset. Show that $f^{-1}(V(I)) = V(f^*(I))$.

b) Let $K = \ker f^* \subseteq A(Y)$. Show that the closure of the image $f(X)$ in Y coincides with $V(K)$.

c) Is it true that f is surjective if and only if f^* is injective? Consider both implications.

d) Is it true that f is injective if and only if f^* is surjective? Consider both implications.

Exercise 4.2 Which of the following ringed spaces are isomorphic over $\mathbb{C}$?

a) $\mathbb{A}^1 \setminus \{1\}$.

b) $V(x_1^2 + x_2^2) \subseteq \mathbb{A}^2$.

c) $V(x_2 - x_1^2, x_3 - x_1^3) \setminus \{0\} \subseteq \mathbb{A}^3$.

d) $V(x_1x_2) \subseteq \mathbb{A}^2$.

e) $V(x_2^2 - x_1^3 - x_1^2) \subseteq \mathbb{A}^2$.

f) $V(x_1^2 - x_2^2 - 1) \subseteq \mathbb{A}^2$

Exercise 4.3 Prove the following statements.

a) Every morphism $\mathbb{A}^1 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^1 \to \mathbb{P}^1$.

b) Not every morphism $\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^1 \to \mathbb{P}^1$.

c) Every morphism $\mathbb{P}^1 \to \mathbb{A}^1$ is constant.

Exercise 4.4 Prove the following statements.

a) Every automorphism of $\mathbb{A}^1$ that fixes the origin is of the form $f(x) = ax$ for one $a \in \mathbb{K}^*$. Hence every automorphism of $\mathbb{A}^1$ is affine.

b) Every automorphism of $\mathbb{A}^2$ that fixes the origin is not of the form $f(x) = Ax$ for one $A \in GL(2, K)$. Hence not every automorphism of $\mathbb{A}^2$ is affine.