Fachbereich Mathematik | Summer Semester 2022, Set 3
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Algebraic Geometry

Submit by: Monday, 09/05/22, 10 am

Exercise 12 a. and b. are an in class exercises. There is no need to hand the solution in.

Exercise 9: Let X be a topological space, $\mathcal{F}$ a sheaf on X and $\varphi, \psi \in \mathcal{F}(U)$ two sections on the open subset $U \subseteq X$ of X.

a. Show, if the stalks $\varphi_a = \overline{(U,\varphi)}$ and $\psi_a = \overline{(U,\psi)}$ in $\mathcal{F}_a$ coincide for all $a \in U$, the $\varphi = \psi$.

b. If X is an irreducible affine variety and $\mathcal{F} = \mathcal{O}_X$ and if there exists one $a \in X$ such that $\varphi_a = \psi_a$, the $\varphi = \psi$.

Exercise 10: Consider $X = \mathbb{R}$ with the euclidean topology and let $a \in X$ be any point. For which of the following sheaves is the stalk $\mathcal{F}_a$ a local ring?

a. $\mathcal{F}$ is the sheaf of continous functions.

b. $\mathcal{F}$ is the sheaf of locally polynomial functions.

Exercise 11: Let X be an affine variety and $Y \subsetneqq X$ be an irreducible subvariety. Show that $A(X)_{I(Y)}$ is isomorphic to the K-algebra $\mathcal{O}_{X,Y}$, where

$$\mathcal{O}_{X,Y} := \{(U,\varphi) \mid U \subseteq X \text{ open with } U \cap Y \neq \emptyset, \varphi \in \mathcal{O}_X(U)\} / \sim$$

with $(U,\varphi) \sim (V,\psi)$ if there is an open $W \subseteq U \cap V$ with $W \cap Y \neq \emptyset$ such that $\varphi_{|W} = \psi_{|W}$.

Exercise 12: Let $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ be polynomials and let $X \subseteq \mathbb{A}^n_K$ an affine variety. Consider the morphisms

$$f : \mathbb{A}^n_K \longrightarrow \mathbb{A}^m_K : p \mapsto \big(f_1(p), \ldots, f_m(p)\big)$$

of affine varieties and

$$\pi : \mathbb{A}^{n+m}_K \longrightarrow \mathbb{A}^m_K : (p_1, \ldots, p_n, q_1, \ldots, q_m) \mapsto (q_1, \ldots, q_m).$$

a. Show that the graph $\mathrm{Graph}(f) = \big\{\big(p, f(p)\big) \bigm| p \in \mathbb{A}^n_K\big\}$ of f is an affine variety in $\mathbb{A}^{n+m}_K$ with

$$I(\mathrm{Graph}(f)) = \langle y_1 - f_1, \ldots, y_m - f_m, I(X)\rangle.$$

b. Show that the image of f satisfies $\mathrm{Im}(f) = \pi(\mathrm{Graph}(f))$ and

$$I(\mathrm{Im}(f)) = I \cap K[y_1, \ldots, y_m].$$

In particular $\overline{\mathrm{Im}(f)} = V(I \cap K[y_1, \ldots, y_m])$.

c. The intersection of an Ideal I as above with the subalgebra $K[y_1, \ldots, y_m]$ can be computed in SINGULAR using the command **eliminate**. If the generating polynomials of I have coefficients over a subfield of K the computations can be done over this subfield. Use SINGULAR to compute the closure of the image of the map

$$f : \mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^3_{\mathbb{C}} : t \mapsto (t, t^2, t^3).$$

The image is called the *twisted cubic*.