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Algebraic Geometry
SS 2022

Exercise Sheet 11

Submit by: Monday, 11/07/22, 10 am

Exercise 11.1 Let $k = \overline{k}$ be an algebraically closed field. Consider the affine plane as the affine scheme $\mathbb{A}^2_k = \operatorname{Spec}(k[x, y])$. Consider the closed subschemes $L := V(y)$ and $C = V(f)$ for some non-constant polynomial $f \in k[x, y]$ with $f \notin \langle y \rangle$.

(a) Find a geometric condition on C, which is necessary and sufficient, for $L \cap C$ to be non-reduced. Hint: start by looking at the case of $\langle x, y \rangle \in L \cap C$.

(b) Use the considerations of (a) to define an "intersection multiplicity $i_P(L, C) \in \mathbb{N}$ of L and C in a closed point P", with the following properties: for all closed points $P \in L$ hold

  (i) $i_P(L, C) = 0$ if and only if $P \notin L \cap C$

  (ii) if C decomposes into distinct components $C = C_1 \cup C_2$, given by non-constant polynomials f_1 and f_2, then $i_P(L, C) = i_P(L, C_1) + i_P(L, C_2)$.

Exercise 11.2 Let $k = \overline{k}$ be an algebraically closed field. Consider the affine scheme given as $X := \operatorname{Spec}(k[x, y]/\langle xy, y^2\rangle)$.

(a) Show that X is irreducible with respect to the Zariski topology.

(b) Show that X is not isomorphic to some variety Y, considered as a scheme as in correspondence (12.39).

(c) Show that X can be viewed as a union of two closed affine subschemes X_1 and X_2 with $X_1, X_2 \neq X$ in the sense of construction (12.28). Explain why this does not contradict (a).

Exercise 11.3 Let $p, q \in \mathbb{N}$ be prime numbers. Consider $X_p := \operatorname{Spec}(\mathbb{Z}_p)$ and $X_q := \operatorname{Spec}(\mathbb{Z}_q)$ as schemes over $\mathbb{Z}$.

(a) Describe the scheme $X_p \times_{\mathbb{Z}} X_q$ over $\mathbb{Z}$.

(b) Give a geometric interpretation of (a).