The functor of points and Yoneda's lemma

An extra exercise after reading Section 1.3 of Vakil

By the classical Nullstellensatz, there is a bijection between radical ideals in $k[x_1, \dots, x_n]$ and Zariski-closed subsets of $\mathbb{A}^n_k$, where k is an algebraically closed field.

What happens if k is not algebraically closed or if the ideals are not radical? Let's start with the non algebraically closed fields. Then this correspondence breaks down: for example, in $\mathbb{R}[x]$ the two radical ideals (x^2+1) and (1) have the same solution set in $\mathbb{A}^1_{\mathbb{R}}$, namely the empty set. To solve this, the point of view that was taken in classical algebraic geometry was that of considering not only the field k but all the field extensions K/k at the same time. For example, in our case of before, the polynomial x^2+1 has coefficients in $\mathbb{R}$, hence also in $\mathbb{C}$, and we can consider the corresponding set of solutions in $\mathbb{A}^1_{\mathbb{C}}$ instead.

More formally, let k be an arbitrary field and let $I \subseteq k[x_1, \dots, x_n]$ be an ideal. We can consider the category Field_k of ring extensions of k, whose objects are field homomorphisms $\varphi\colon k \to K$ (what are the morphisms?). Then for any $f \in k[x_1, \dots, x_n]$ we obtain a polynomial $\varphi(f) \in K[x_1, \dots, x_n]$ by applying φ to the coefficients of f. Then we can define the covariant *functor of points*

$$\begin{aligned} \mathbb{V}(I)\colon\ & \text{Field}_k \longrightarrow \text{Sets} \\ & (\varphi\colon k \to K) \mapsto \mathbb{V}(I)(K) := \{(a_1, \dots, a_n) \in K^n \mid \varphi(f)(a_1, \dots, a_n) = 0 \text{ for all } f \in I\} \end{aligned}$$

Exercise 1.

a) Show that this is a functor.

b) Show that for two ideals $I, J \subseteq k[x_1, \dots, x_n]$ the two functors $\mathbb{V}(I)$ and $\mathbb{V}(J)$ coincide if and only if $\sqrt{I} = \sqrt{J}$ in $k[x_1, \dots, x_n]$.

However, even if we work over an algebraically closed field, we still have the issue with radical ideals. For example, the two ideals (x) and (x^2) of $\mathbb{C}[x]$ are distinct, but define the same solution set in $\mathbb{A}^1_{\mathbb{C}}$. It turns out that this can be again solved by the functor of points approach, if we use all rings, instead of just fields. More precisely let us consider an arbitrary ring R and an ideal $I \subseteq R[x_1, \dots, x_n]$. We can consider the category Ring_R of ring extensions of R, or R-algebras, whose objects are ring homomorphisms $\varphi\colon R \to S$ (what are the morphisms?). Then for any $f \in R[x_1, \dots, x_n]$ we obtain a polynomial $\varphi(f) \in S[x_1, \dots, x_n]$ by applying φ to the coefficients of f. Then we can define the covariant *functor of points*:

$$\begin{aligned} \mathbb{V}(I)\colon\ & \text{Ring}_R \longrightarrow \text{Sets} \\ & (\varphi\colon R \to S) \mapsto \mathbb{V}(I)(S) := \{(a_1, \dots, a_n) \in S^n \mid \varphi(f)(a_1, \dots, a_n) = 0 \text{ for all } f \in I\} \end{aligned}$$

Exercise 2.

a) Show that this is a functor.

b) Show that for two ideals $I, J \subseteq R[x_1, \dots, x_n]$ the two functors $\mathbb{V}(I)$ and $\mathbb{V}(J)$ coincide if and only if $I = J$ in $R[x_1, \dots, x_n]$. *Hint:* one way of proving it is by looking at points (c) and (d) below.

c) Observe that if we have an ideal $I \subseteq R[x_1, \ldots, x_n]$ we get automatically a ring extension of R by $R \to R[x_1, \ldots, x_n]/I$. Then we can consider the covariant functor

$$h^{R[x_1,\ldots,x_n]/I} \colon \mathrm{Ring}_R \to \mathrm{Sets}, \qquad S \mapsto \mathrm{Mor}(R[x_1, \ldots, x_n]/I, S)$$

as in Section 1.3.10 of Vakil. Show that for any R-algebra S there are canonical isomorphisms

$$\mathbb{V}(I)(S) \cong h^{R[x_1,\ldots,x_n]/I}(S).$$

More precisely, show that there is a natural isomorphism of functors $\mathbb{V}(I) \cong h^{R[x_1,\ldots,x_n]/I}$.

d) Deduce point (b) from Yoneda's Lemma in Section 1.3.10 of Vakil.

In particular, observe that this generalization to arbitrary rings is essentially a formal statement, so in a sense it is much easier than the statement in Exercise 1 for fields.