FOUNDATIONS of ALGEBRAIC GEOMETRY

Algebraic geometry studies solutions to polynomial systems of equations.

Classically, we consider a field k and a collection of polynomials

$$f_1, \dots, f_n \in k[x_1, \dots, x_n]$$

Then these define a variety $X \subseteq k^n$ as

$$X = \mathbb{V}(f_1, \dots, f_n) = \{ (a_1, \dots, a_n) \in k^n \mid f_i(a_1, \dots, a_n) = 0 \}$$

The variety X does not change if we pass to the ideal $I = (f_1, \dots, f_n)$ generated by the $f_1, \dots, f_n$. So we can associate a variety $\mathbb{V}(I)$ to any ideal $I \subseteq k[x_1, \dots, x_n]$.

Conversely, if $X \subseteq k^n$ is a subvariety, we obtain an ideal

$$\mathbb{I}(X) = \{ f \in k[x_1, \dots, x_n] \mid f(p) = 0 \quad \forall p \in X \}$$

The key result of classical algebraic geometry is the NULLSTELLENSATZ, that establishes a dictionary between varieties and ideals:

<u>Thm</u>: HILBERT'S NULLSTELLENSATZ

Let k be an algebraically closed field and $I \subseteq k[x_1, \dots, x_n]$ an ideal. Then

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}.$$

In particular, this establishes a correspondence

$$\left\{ \begin{array}{c} \text{subvarieties of} \\ k^n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{radical ideals} \\ \text{in } k[x_1, \dots, x_n] \end{array} \right\}$$

whenever k is algebraically closed. Hence, we can study geometric questions with algebraic methods and viceversa.

A somehow "dual" version of the above correspondence is given by associating to each variety $X \subseteq k^n$ the quotient ring $k[X] = k[x_1, \dots, x_n] / \mathbb{I}(X)$

Then this can be seen as the ring of polynomial functions on X. Indeed if $F, G \in k[x_1, \dots, x_n]$ are two polynomials and $F, G : k^n \to k$ the corresponding functions, we can restrict to X and obtain two functions

$$F_{|X}, G_{|X} : X \longrightarrow k$$

Then, by definition, we see that

$$F_{|X} = G_{|X} \iff F \equiv G \mod I \iff F = G \text{ in } k[X]$$

<u>Example</u>: We take $(x) \subseteq \mathbb{C}[x]$. Then $\mathbb{V}(x) = \{0\}$ and $\mathbb{C}[x]/(x) = \mathbb{C}$ is the ring of polynomial functions restricted to $\{0\}$. Since $\{0\}$ is just a point these are constants.

This point of view is particularly useful because it abstracts the variety X from the particular embedding $X \subseteq k^n$. Indeed, suppose that $X \subseteq k^n$, $Y \subseteq k^m$ are two varieties and suppose that the two rings

$$k[X] = k[x_1, \dots, x_n]/\mathbb{I}(X) \cong k[y_1, \dots, y_m]/\mathbb{I}(Y) = k[Y]$$

are isomorphic as k-algebras. Then

<u>Exercise</u>: Show that there are polynomial maps $F : k^n \to k^m$, $G : k^m \to k^n$ that restrict to isomorphisms $X \cong Y$.

So, in a way, we can look at the ring $A = k[X]$ as determining the ~~the~~ of the variety X without mention of the embedding $X \hookrightarrow k^n$. This yields a correspondence

$$\left\{ \begin{matrix} \text{affine} \\ k\text{-varieties} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{reduced and f.g.} \\ k\text{-algebras} \end{matrix} \right\}$$

$$X \longmapsto k[X]$$

<u>Example</u>: The field $\mathbb{C}$ corresponds to a point.

$$\{pt\} \longleftrightarrow \mathbb{C}$$

- $\{pt\} \hookrightarrow \{0\}$ in $\mathbb{C}^1$: $\mathbb{C} \cong \mathbb{C}[x]/\mathbb{I}(0) = \mathbb{C}[x]/(x)$
- $\{pt\} \hookrightarrow \{(1,2)\}$ in $\mathbb{C}^2$: $\mathbb{C} \cong \mathbb{C}[x,y]/\mathbb{I}(1,2) = \mathbb{C}[x,y]/(x-1, y-2)$

At this point we make some remarks:

(1) Everything here is about affine varieties. However there are many interesting varieties that are not affine. For example the projective space. We know how to deal with it: we glue together various affine pieces, so it is not too bad.

Two things are more serious though:

(2) The hypothesis of being algebraically closed is essential.

Example: For the two radical ideals (x^2+1) and (1) of $\mathbb{R}[x]$ we have

$$\mathbb{V}(x^2+1) = \mathbb{V}(1) = \emptyset \text{ in } \mathbb{R}^1$$

but the ideals are definitely distinct.

Example: For the two radical ideals (x^3-x) and (0) in $\mathbb{F}_3[x]$ we have

$$\mathbb{V}(x^3-x) = \mathbb{V}(0) = \mathbb{F}_3^1$$

but the ideals are distinct.

(3) The correspondence only sees radical ideals, so that nilpotents are canceled.

<u>Example</u>: The ideals (x) and (x^2) in $\mathbb{C}[x]$ define the same set $\mathbb{V}(x) = \mathbb{V}(x^2) = \{0\}$ in $\mathbb{C}^1$. However, if we look at the corresponding rings

$$\mathbb{C}[x]/(x) = \mathbb{C} \qquad \mathbb{C}[x]/(x^2) = \mathbb{C}[\varepsilon] \qquad \varepsilon^2 = 0$$

we can look at the first as being given by the evaluation of a polynomial at 0:

$$\begin{aligned} \mathbb{C}[x] &\longrightarrow \mathbb{C}[x]/(x) \cong \mathbb{C} \\ f(x) &\longmapsto f(0) \end{aligned}$$

whereas the second gives the evaluation up to the first order

$$\begin{aligned} \mathbb{C}[x] &\longrightarrow \mathbb{C}[x]/(x^2) \cong \mathbb{C}[\varepsilon] \\ f(x) &\longmapsto f(0) + f'(0)\varepsilon \end{aligned}$$

So if we want a parallel as before, we should expect

$(x) \leftrightsquigarrow \mathbb{C}[x]/(x) \leftrightsquigarrow$ 0-th order information at $\{0\}$ $\leftrightsquigarrow$ point

$(x^2) \leftrightsquigarrow \mathbb{C}[x]/(x^2) \leftrightsquigarrow$ 1-st order information at $\{0\}$ $\leftrightsquigarrow$ fat point

However, this is lost in the previous correspondence.

- SCHEMES

These issues are solved in Grothendieck's theory of schemes, which accomplishes three important generalizations at the same time, connecting algebraic geometry with three other distinct areas of mathematics.

(1) GEOMETRY : one of the key ideas of modern (= after Riemann) geometry is that of a MANIFOLD. This is a global space obtained by gluing together local pieces that we understand well.

- DIFFERENTIABLE MANIFOLD : space obtained by gluing together Euclidean balls in $\mathbb{R}^n$ along differentiable maps.
  - Locally described by ANALYSIS

- COMPLEX MANIFOLD : space obtained by gluing together Euclidean balls in $\mathbb{C}^n$ along holomorphic maps.
  - Locally described by COMPLEX ANALYSIS

- SCHEME : space obtained by gluing together affine schemes.
  - Locally described by COMMUTATIVE ALGEBRA

We have seen this idea already before: in classical algebraic geometry, the building blocks are affine varieties in k^n and many other varieties, such as $\mathbb{P}^n$, can be obtained by gluing these pieces together.

This is vastly extended by the language of schemes where the building blocks are the so-called affine schemes.

Another general principle of modern geometry is that spaces can be understood in terms of functions on them. We already saw this for affine varieties over $k = \bar{k}$. It will be generalized and made precise in the language of sheaves.

(2) NUMBER THEORY: We don't just at equations over algebraically closed fields, but over general fields, and even general rings. This is basically number theory, that looks at solutions of equations over $\mathbb{Q}$ or $\mathbb{Z}$, ...

(3) ANALYSIS: We allow nilpotents, so we will actually have a concrete space associated with the ring $\mathbb{C}[\varepsilon] = \mathbb{C}[x]/(x^2)$, and this space will be indeed a "fat point". So we can take approximation of functions, which is basically analysis.