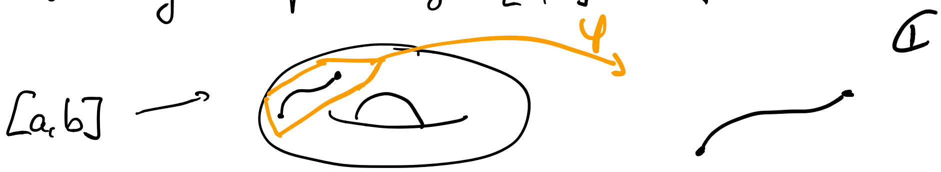
ABEL'S THEOREM

① Integration on Riemann surfaces.

Let ω be a holomorphic differential on a compact Riemann surface X

Locally: $\omega = f(z)\,dz$

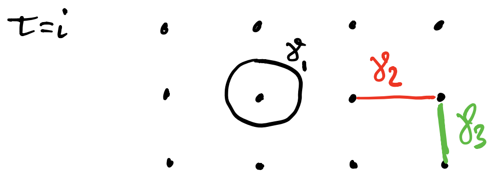
Integrate ω along a path $\gamma : [a,b] \to X$



$$\int_\gamma \omega = \int_a^b f(\varphi(\gamma(t))) \cdot (\varphi \circ \gamma)'(t)\, dt \in \mathbb{C}$$

$z = \varphi(\gamma(t))$

Exm: $X = \mathbb{C}/L$, $L = \mathbb{Z} \oplus \mathbb{Z}\tau$, $\mathrm{Im}(\tau) > 0$

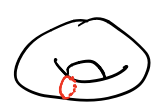


$\gamma_1 : [0, 2\pi] \to \mathbb{C}$

$\gamma_1(t) = \frac{1}{2}(\cos(t), \sin(t)) = \frac{1}{2} e^{it}$

$\omega = dz$

$$\int_{\gamma_1} \omega = \int_0^{2\pi} i \tfrac{1}{2} e^{it}\, dt = \left[\tfrac{1}{2} e^{it}\right]_0^{2\pi} = 0$$



$\gamma_2 : [0,1] \to \mathbb{C}$, $\gamma_2(t) = 1 + t$

$$\int_{\gamma_2} \omega = \int_0^1 1\, dt = 1$$

$$\int_{\gamma_3} \omega = i$$

FACT: The integral $\int_\gamma \omega$ of ω along a closed path $\gamma : [0,1] \to X$ $(\gamma(0) = \gamma(1))$ depends only on the homology class of γ.

map
$$\mathbb{Z}^{2g} \cong H_1(X,\mathbb{Z}) \longrightarrow H^0(X,\omega_0)^* \cong \mathbb{C}^g$$
$$[\gamma] \longmapsto \int_\gamma \omega$$

is a <u>group homomorphism</u>
$$\int_{\gamma_1+\gamma_2}\omega = \int_{\gamma_1}\omega + \int_{\gamma_2}\omega$$

<u>Def:</u> A linear functional $\lambda : H^0(X,\omega) \longrightarrow \mathbb{C}$ is a <u>PERIOD</u> if $\lambda = \int_{[\gamma]} \omega$ for some $[\gamma] \in H_1(X,\mathbb{Z})$.

<u>FACT:</u> The periods form a <u>lattice</u> in $H^0(X,\omega)^*$.

<u>Def:</u> The JACOBIAN of X is the quotient of $H^0(X,\omega)^*$ modulo the subgroup Λ of periods:
$$\mathrm{Jac}(X) = H^0(X,\omega)^*/\Lambda \underset{\text{as a group!}}{\cong} \mathbb{C}^g/\mathbb{Z}^{2g}$$

<u>Exm:</u> a) $X \cong \mathbb{P}^1$: $\mathrm{Jac}(X) = \{0\}$

b) $X = \mathbb{C}/L$: $\mathrm{Jac}(X) \cong X$

ABEL-JACOBI MAP

Fix a base point $P_0 \in X$. For any $P \in X$, choose a path γ_P on X from P_0 to P.

We get a map
$$A : X \to H^0(X,\omega)^*$$
$$A(P) \mapsto \left(\omega \longmapsto \int_{\gamma_P}\omega\right)$$

This map depends on the choice of P_0 and the choice of the path γ_P from P_0 to P.

It depends on the choice of γ_P only up to periods!

$\gamma_P - \gamma'_P$ is a closed path, so

$\int_{\gamma_P - \gamma'_P} \omega$ is a period.

So we get a map

$$\boxed{\begin{aligned} A: \quad & X \to \mathrm{Jac}(X) \\ & A(P)(\omega) = \int_{\gamma_P} \omega \end{aligned}}$$

ABEL-JACOBI MAP

(still depends on the choice of P_0)

Choosing a basis $\omega_1, \dots, \omega_g$ of $H^0(X, \omega)$, this map is $A(P) = \left(\int_{\gamma_P} \omega_1, \dots, \int_{\gamma_P} \omega_g \right) \in \mathbb{C}^g/\Lambda$

The ABEL-JACOBI MAP extends linearly to divisors on X:

$$A: \mathrm{Div}(X) \longrightarrow \mathrm{Jac}(X)$$

$$\sum n_P P \longmapsto \underbrace{\sum n_P A(P)}_{\text{sum as elements of the group } \mathrm{Jac}(X)}$$

$A_0: \mathrm{Div}_0(X) \to \mathrm{Jac}(X)$: map induced on divisors of degree 0.

<u>Prop:</u> The map $A_0: \mathrm{Div}_0(X) \to \mathrm{Jac}(X)$ does not depend on the choice of the base point $P_0 \in X$.

<u>Proof:</u> Let P_0' be another point in X and choose a path γ from P_0 to P_0'.

$$A(P) = \left(\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_g \right)$$

$$A'(P) = \left(\int_{P_0'}^{P} \omega_1, \dots, \int_{P_0'}^{P} \omega_g \right) =$$

$$\overset{\text{in } Jac(X)}{=} \underbrace{\left(\int_{P_0'}^{P_0} \omega_1, \dots, \int_{P_0'}^{P_0} \omega_g \right)}_{j :=} + \underbrace{\left(\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_g \right)}_{A(P)}$$

$$A'(\Sigma n_P \cdot P) = \underbrace{(\Sigma n_P)}_{=0} j + A(\Sigma n_P \cdot P) \qquad \square$$

<u>THM:</u> (ABEL)

Let D be a divisor of degree 0 on X.

Then D is a principal divisor on X if and only if

$$A_0(P) = 0 \quad \text{in } Jac(X).$$

<u>Exm:</u> a) $X \cong \mathbb{P}^1$, $Jac(X) = \{0\}$: Every divisor of degree 0 is principal ✓.

b) $X \cong \mathbb{C}/L$, $Jac(X) \cong X$: $A_0(\Sigma n_P \cdot P)$ is $\Sigma n_P \cdot P$ (group law on X)

<u>Cor:</u> X has genus ≥ 1, Then $A : X \to Jac(X)$ is an embedding (but depends on the choice of base point P_0).

<u>Proof:</u> A is a holomorphic map of complex manifolds. It is also injective: If $A(P) = A(P')$, then

$$A(P-P') = 0 \overset{\text{ABEL}}{\Longrightarrow} P-P' \text{ is principal}$$

$$\Rightarrow X \cong \mathbb{P}^1. \quad \text{if} \quad P \neq P'$$

$$\Rightarrow P = P'. \qquad \square$$

<u>SCHOTTKY PROBLEM.</u> Which $\mathbb{C}^g/\Lambda$ are Jacobians of curves?

$$\exp\left((z_1, \dots, z_g)(A+iB)\begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix} \right)$$