PROJECTIVE GEOMETRY OF CURVES

CANONICAL CURVES.

Let X be an alg. curve (compact (connected) Riemann surface) and $K = (\omega)$ a canonical divisor

Prop: Either $|K|$ is very ample or X is hyperelliptic

Remark: If $|K|$ is very ample, then the genus of X is at least 3.

Def: A canonical curve is an alg. curve $X \overset{j}{\hookrightarrow} \mathbb{P}^{g-1}$ that is not hyperelliptic and such that a hyperplane section $\mathrm{div}_X(H) = j^*(H)$ is a canonical divisor of X.

Def: The degree $\deg(X)$ of a nondegenerate alg. curve $X \subset \mathbb{P}^n$ is the degree of the linear system $|\mathrm{div}_X(H)|$, where $H \subseteq \mathbb{P}^n$ is a hyperplane.

Prop: A canonical curve $X \subseteq \mathbb{P}^{g-1}$ has degree $2g-2$.

Classification of curves of low genus.

① $g=0$: $X \cong \mathbb{P}^1$

② $g=1$: $X \cong \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ for a $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau)>0$

$X \cong$ smooth plane cubic

③ $g=2$: X is hyperelliptic : affine model $\{y^2 = f(x)\}$ for a polynomial f of degree 6 with distinct roots

④ $g=3$:

Case 1: X is hyperelliptic

Case 2: $\phi_K : X \hookrightarrow \mathbb{P}^2$ is an embedding.

$\phi_K(X) \subseteq \mathbb{P}^2$ has degree 4

What is the ideal of $\phi_K(X)$ in $\mathbb{C}[x_1, x_2, x_3]$? $\qquad \mathbb{P}^2 = \{(x_1, x_2, x_3)\}$

For every degree d fix a homogeneous polynomial F_0 in $\mathbb{C}[x_1, \dots, x_g]$ (e.g. $F_d = F_1^d$):

We get a map $R_d : \mathbb{C}[x_1, \dots, x_g]_d \longrightarrow H^0(X, d \cdot K)$

$$F \longmapsto \frac{F}{F_d}$$

$$\operatorname{div}\left(\frac{F}{F_d}\right) = \operatorname{div}(F) - \underbrace{\operatorname{div}(F_d)}_{= d \cdot K} = \operatorname{div}(F) - dK$$

Dimension counting: $\quad \dim \mathbb{C}[x_1, \dots, x_g]_d = \binom{g+d-1}{d}$

$$\dim H^0(X, dK) \underset{d \geq 2}{=} d(2g-2) + 1 - g = (2d-1)(g-1)$$

<u>$g=3$: $d=4$.</u> $\quad h^0(X, 4K) = 2 \cdot 7 = 14$

$$\dim(\mathbb{C}[x_1, x_2, x_3]_4) = \binom{6}{2} = 15$$

<u>Prop:</u> A curve of genus 3 is either hyperelliptic or a smooth quartic plane curve.

(I) <u>$g=4$:</u> Case 1: X is hyperelliptic.

Case 2: $\phi_K : X \hookrightarrow \mathbb{P}^3$, curve of degree 6.

$\dim \ker(R_2 : \mathbb{C}[x_1, \dots, x_4]_2 \to H^0(X, 2K)) \geq 10 - 3 \cdot 3 = 1$. $\Rightarrow$ There is a quadric Q in $I(X)$

$\dim \ker(R_3 : \mathbb{C}[x_1, \dots, x_4]_3 \to H^0(X, 3K)) \geq \binom{6}{3} - 5 \cdot 3 = 20 - 15 = 5$

$\Rightarrow$ There is also a cubic C in $I(X)$ that is not a multiple of Q

$\rightsquigarrow$ $I(X) = \langle Q, C \rangle$

<u>Prop:</u> A curve of genus 4 is hyperelliptic or the complete intersection of a quadric and a cubic in $\mathbb{P}^3$

GEOMETRIC FORM OF RIEMANN-ROCH

Let $X \hookrightarrow \mathbb{P}^{g-1}$ be a canonical curve.

Let $D = P_1 + \dots + P_d$ be an effective divisor of degree d on X

If the points are distinct, then span(D) is the smallest linear space in $\mathbb{P}^{g-1}$ that contains $P_1, \dots, P_d$.

Def: A hyperplane $H \subseteq \mathbb{P}^{g-1}$ 'contains a divisor D' if

$$\mathrm{div}_X(H) \geq D.$$

H contains $P \in X \iff P \in H$

The linear system of hyperplanes in $\mathbb{P}^{g-1}$ that contain D is isomorphic to $|K-D|$

Def: The span of a divisor on $X \hookrightarrow \mathbb{P}^{g-1}$ is the intersection of all hyperplanes $H \subseteq \mathbb{P}^{g-1}$ that contain D.

By definition of the span of a divisor D, we have

$$\dim(\mathrm{span}(D)) + h^0(X, K-D) = g-1$$

$$\dim(\underset{\subseteq \mathbb{C}^g}{\widehat{\mathrm{span}(D)}}) + h^0(X, K-D) = g$$

$$\mathbb{P}(\widehat{\mathrm{span}(D)}) = \mathrm{span}(D)$$

Thm: Let $X \hookrightarrow \mathbb{P}^{g-1}$ be a canonical curve.

For any divisor D on X we have

$$\dim |D| = \deg(D) - 1 - \dim \mathrm{span}(D)$$

Proof:

$$h^0(X,D) \cancel{-1} = \deg(D) \cancel{-1} - (g-1-h^0(X,K-D))$$

$$h^0(X,D) = \deg(D) - g + 1 + h^0(X, K-D) \qquad \square$$

(6) $g=5$: $\phi_K : X \hookrightarrow \mathbb{P}^4$

$$R_2 : \mathbb{C}[x_0, \dots, x_4]_2 \longrightarrow H^0(X, 2K)$$

dim: $\quad 15 \qquad\qquad 3\cdot 4$

$\Rightarrow \dim(\ker(R_2)) \geq 3$

Can 3 points on X be collinear? Take $P_1, P_2, P_3 \in X$, $D = P_1 + P_2 + P_3$.

$\Rightarrow \dim |D| = 3 - 1 - \underbrace{\dim \operatorname{span}(D)}_{\text{if this is } 1,\text{ then } \dim |D| = 1}$

Suppose such three points exist, then $h^0(D) = 2$.

Then there is a 1-dimensional family of lines that intersect X in 3 points and their union is a surface: rational normal scroll.

Such canonical curves of genus 5 are called <u>trigonal</u>.

<u>CAYLEY-BACHARACH.</u>

<u>Thm</u>: Let $X_1, X_2 \subseteq \mathbb{P}^2$ be two curves of degree d and e respectively.
Suppose $d \geq e$ and X_1 is smooth.
Suppose that $X_1 \cap X_2$ consists of $d\cdot e$ many distinct points.
$P_1, \dots, P_{d\cdot e}$
If $C \subseteq \mathbb{P}^2$ is any curve of degree $d+e-3$ containing all but one point of $X_1 \cap X_2$, then it contains all points of $X_1 \cap X_2$.

<u>Fact 1</u>: Let H be the intersection divisor of a line $L \subseteq \mathbb{P}^2$ with X_1. Then $K_{X_1} \sim (d-3)\cdot H$

<u>Cor (Riemann-Roch)</u>: Every effective divisor on X that is linearly equivalent to $(d-3)\cdot H + P$ for $P \in X$ actually contains P.

Proof of Cayley-Bacharach:

$$\mathrm{div}_{X_1}(C) = C.X_1 = P_1 + \ldots + P_{d\cdot e-1} + Q_1 + \ldots + Q_{d\cdot(d-3)+1}$$

$$[(d+e-3)\cdot d = d\cdot e - 1 + d\cdot(d-3)+1$$

$$K_{X_1} \sim (d-3)\cdot H \quad \text{and} \quad P_1 + \ldots + P_{d\cdot e} \sim e\cdot H$$

$$C.X_1 \sim (d+e-3)\cdot H$$

$$\Rightarrow (d+e-3)\cdot H \sim e\cdot H - P_{d\cdot e} + Q_1 + \ldots + Q_{d(d-3)+1}$$

$$\Rightarrow (d-3)\cdot H \sim Q_1 + \ldots + Q_{d(d-3)+1} - P_{de}$$

$$(d-3)\cdot H + P_{d\cdot e} \sim Q_1 + \ldots + Q_{d(d-3)+1}$$

$$\Rightarrow P_{d\cdot e} \in \{Q_1, \ldots, Q_{d(d-3)+1}\} \qquad \square$$