# § 9 : DIFFERENTIAL FORMS and RIEMANN-ROCH

We shift gears for a bit.

First, let $U \subseteq \mathbb{C}$ be an open subset. An HOLOMORPHIC DIFFERENTIAL FORM $\omega$ on $U$ has the form

$$\omega = f(z)\,dz \quad \text{where } f \text{ is a holomorphic function.}$$

When $f$ is meromorphic instead, we talk of a MEROMORPHIC DIFFERENTIAL FORM.

This notion makes sense also on any Riemann surface $X$. An HOLOMORPHIC FORM $\omega$ on $X$ can be described by a collection of holomorphic forms

$$\omega_i = f_i(z_i)\,dz_i \quad \text{on each chart } U_i$$

which coincide on the intersections $U_i \cap U_j$. The same for meromorphic differential forms.

The space of all holomorphic forms on a Riemann surface is a complex vector space that we denote by $H^0(X, \omega_X)$, or $H^0(X, \Omega^1_X)$.

Example: (1) HOLOMORPHIC FORMS on $\mathbb{P}^1$

Take the two usual charts $U_0, U_1$ of $\mathbb{P}^1$ with affine coordinates $x$ and $z$ respectively, s.t. $z = 1/x$ on $U_0 \cap U_1$. Then a holomorphic form is

$$\omega_0 = f_0(x)\,dx \qquad f_0 \text{ holomorphic}$$
$$\omega_1 = f_1(z)\,dz \qquad f_1 \text{ holomorphic}$$

such that $\omega_0|_{U_0\cap U_1} = \omega_1|_{U_0 \cap U_1}$. On this intersection we see

$$\omega_0|_{U_0\cap U_1} = f_0(x)\,dx = f_0\left(\frac{1}{z}\right) d\left(\frac{1}{z}\right)$$
$$= -\frac{f_0(1/z)}{z^2}\,dz$$

So the condition is that $f_1(z) = -\frac{1}{z^2} \cdot f_0\left(\frac{1}{z}\right)$

However this is impossible because the expression on the right has always a pole at $z=0$.

Hence, on $\mathbb{P}^1$ the only holomorphic differential form is the zero form.

$$H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}) = 0$$

(2) HOLOMORPHIC FORMS on a COMPLEX TORUS

Take a complex torus $X_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ as usual.

Then on $\mathbb{C}$ we have the differential form $dz$ which is invariant under the action of the lattice

$$d(z+m+\tau n) = dz$$

Then one can see that this induces a nonzero holomorphic form $dz$ on $X$. Furthermore this is essentially the only one; morally, if $\omega$ is a differential form on $\mathbb{C}$ then it must have the form

$$\omega = f(z)\,dz$$

with $f$ holomorphic on $\mathbb{C}$ and s.t. $f(z+m+\tau n) = f(z)$, but any such $f$ is constant. Hence

$$H^0(X_\tau, \omega_{X_\tau}) = \mathbb{C}\cdot dz$$

(3) HOLOMORPHIC FORMS on PLANE CURVES

Suppose $X = \{f(x,y)=0\}$ is a smooth plane curve and set $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$.

On the set $U_1 = \{f_x \neq 0\}$ we can define the form

$$\omega_1 = \frac{1}{f_x}\,dy$$

Can we extend this to a form also on the chart $U_2 = \{f_y \neq 0\}$? Morally we do it like this: since $f(x,y) = 0$ on $X$, we have that

$$f_x \, dx + f_y \, dy = 0$$

so

$$1/f_x \, dy = - 1/f_y \, dx$$

and this extends $\omega_1$ to a form $\omega$ on the whole of $X$. As an exercise, make this rigorous.

- FORMS and DIVISORS

Let $\omega$ be a meromorphic differential form. Then on a collection of charts $U_i$ we can write $\omega_{|U_i} = f_i(z_i) dz_i$ for $f_i(z_i)$ meromorphic on $U_i$. We then define

$$\boxed{\operatorname{ord}_p(\omega) := \operatorname{ord}_p f_i} \quad \text{if } p \in U_i$$

This is well defined: if $p \in U_i \cap U_j$ then

$$f_i(z_i) dz_i = f_j(z_j) dz_j \quad \text{on } U_i \cap U_j$$

Let $z_i = \varphi(z_j)$ be the change of coordinate. Then

$$f_i(z_i)dz_i = (f_i\circ\varphi)(z_j)\,d\varphi(z_j)$$
$$= (f_i\circ\varphi)(z_j)\cdot\dot{\varphi}(z_j)\,dz_j$$

hence $f_j(z_j) = (f_i\circ\varphi)(z_j)\,\dot{\varphi}(z_j)$

and since $\dot{\varphi}(z_j)\neq 0$, the two orders coincide.
The same argument works when we choose another collection of representatives for $\omega$.

<u>Def</u>: CANONICAL DIVISOR
For any meromorphic differential form $\omega$ we can define the divisor

$$\boxed{\operatorname{div}(\omega) = \sum_{p\in X} \operatorname{ord}_p(\omega)\cdot p}$$

and any divisor of this form is called a canonical divisor on $X$, sometimes denoted by $K$

<u>Facts</u>: (1) Any two canonical divisors are linearly equivalent.

<u>proof</u>: Let $\omega_1, \omega_2$ be two meromorphic forms
Then on a collection of charts $U_i$ we have:
$$\omega_{1|U_i} = f_{1,i}(z_i)\,dz_i$$
$$\omega_{2|U_i} = f_{2,i}(z_i)\,dz_i$$

We claim that the collection $\left(\frac{f_{1,i}(z_i)}{f_{2,i}(z_i)}\right)$ is a meromorphic function on $X$.

Indeed, on $U_i \cap U_j$ let $z_i = \varphi(z_j)$ be the change of coordinates. Then

$$f_{1,j}(z_j) = f_{1,i}(\varphi(z_j))\,\dot{\varphi}(z_j)$$
$$f_{2,j}(z_j) = f_{2,i}(\varphi(z_j))\,\dot{\varphi}(z_j)$$

so
$$\frac{f_{1,j}(z_j)}{f_{2,j}(z_j)} = \frac{f_{1,i}(z_i)\cdot\cancel{\dot{\varphi}(z_j)}}{f_{2,i}(z_i)\,\cancel{\dot{\varphi}(z_j)}} = \frac{f_{1,i}(z_i)}{f_{2,i}(z_i)}.$$

We denote this meromorphic function by

$$\left(\frac{\omega_1}{\omega_2}\right)$$

Then it is easy to see that

$$\operatorname{div}(\omega_1) - \operatorname{div}(\omega_2) = \operatorname{div}(f). \qquad \square$$

(2) If $K$ is any canonical divisor, then

$$H^0(X, \omega_X) \cong H^0(X, K)$$

proof: suppose $K = \operatorname{div}(\omega_0)$ for a meromorphic differential $\omega_0$. Then for any $\omega \in H^0(X, \omega_X)$ we see that

$$\operatorname{div}\left(\frac{\omega}{\omega_0}\right) = \operatorname{div}(\omega) - \operatorname{div}(\omega_0)$$
$$= \operatorname{div}(\omega) - K \geqslant -K$$

because $\operatorname{div}(\omega)$ is effective. Hence we have

$$H^0(X, \omega_X) \longrightarrow H^0(X, K)$$
$$\omega \longmapsto \left(\frac{\omega}{\omega_0}\right)$$

We also have the inverse map: if $f \in H^0(X, K)$ we can define a meromorphic differential $f \cdot \omega_0$ in a collection of charts $U_i$ as

$$f \cdot \omega_0|_{U_i} = f_{|U_i}(z_i)\, \omega_{0|U_i}$$

As an exercise, show that the map

$$H^0(X, K) \longrightarrow H^0(X, \omega_X)$$
$$f \longmapsto f \cdot \omega_0$$

is well-defined and an inverse of the previous one.

(3) More generally, for any divisor $D$ on $X$ we can define

$$H^0(X, \omega_X(D)) = \left\{ \omega \ \begin{smallmatrix}\text{merom.}\\ \text{diff.}\end{smallmatrix} \ \middle| \ \mathrm{div}(\omega) \geq -D \right\}$$

The same proof of before shows that

$$H^0(X, \omega_X(D)) \cong H^0(X, K_X + D)$$

where $K_X$ is any canonical divisor on $X$.

- PULLBACKS of FORMS

We can pull back differentials along maps of Riemann surfaces.

Locally, consider a map $F : U \to V$ with $U, V \subseteq \mathbb{C}$ open sets.
Let $\omega = f(w)dw$ be a meromorphic diff. on $V$

$$F^*\omega := f(F(z))\, dF(z) = f(F(z))\, F'(z)\, dz$$

and this generalizes via charts to maps $F : X \to Y$ of Riemann surfaces.
We can interpret the Riemann-Hurwitz theorem in this way. Let $F : X \to Y$ be a map of compact Riemann surfaces. Recall:

- If $q \in Y$, we define the divisor

$$F^*q = \sum_{p \in F^{-1}(q)} \mathrm{mult}_p(F) \cdot p$$

If $D = m_1 q_1 + \dots + m_r q_r$ is any divisor on $Y$ we define $F^*D = m_1 F^*q_1 + \dots + m_r F^*q_r$. Observe that $\deg(F^*D) = \deg F \cdot \deg D$.

- The ramification divisor of $F$ is $R = \sum_{p \in X} (\mathrm{mult}_p(F) - 1) \cdot p$

Thm: [ RIEMANN-HURWITZ II ]

Let $F: X \to Y$ be a map of compact Riemann surfaces of degree $d$. Let $\omega$ be a meromorphic differential on $Y$. Then

$$\mathrm{div}(F^*\omega) = F^*(\mathrm{div}\,\omega) + R$$

Proof: Let $p \in X$ and $q = f(p) \in Y$.
Locally around $p$ and $q$ the map has the shape $F: U \to V$, $F(z) = z^m$, where $m = \mathrm{mult}_p F$. If $\omega = f(w)dw$ on $V$ then
$F^*\omega = f(F(z))dz^m = m \cdot f(z^m)) \cdot z^{m-1} dz$.
Let $e = \mathrm{ord}_q(\omega) = \mathrm{ord}_q f(w)$. Then
$f(w) = a \cdot w^e + \dots$ and $f(z^m) = a \cdot z^{me} + \dots$
hence $\mathrm{ord}_q f(z^m) = m \cdot e$. In conclusion

$$\begin{aligned}\mathrm{ord}_p F^*\omega &= \mathrm{ord}_p(f(z^m)) + \mathrm{ord}_p(z^{m-1}) \\ &= m \cdot \mathrm{ord}_q(f(w)) + (m-1) \\ &= \mathrm{ord}_p(F^*(\mathrm{div}\,\omega)) + \mathrm{ord}_p R\end{aligned}$$

which proves what we wanted. $\square$

Rmk: We can write this as $K_X = F^* K_Y + R$.

- RIEMANN-ROCH

We can finally state one of the most important results in the theory of Riemann surfaces and algebraic curves.

**Thm**: RIEMANN-ROCH

Let $X$ be a compact Riemann surface of genus $g$ and $D$ a divisor on $X$.

$$\boxed{h^0(X,D) - h^0(X,\omega_X(-D)) = \deg D + 1 - g}$$

This has a lot of consequences. Let's see some.

(1) If $\deg D \geq g-1+n$, then $h^0(X,D) \geq n$.

proof: $h^0(X,D) = \deg D + 1 - g + h^0(X,\omega_X(-D))$
$$\geq n + h^0(X,\omega_X(-D)) \geq n. \qquad \square$$

(2) $X$ has a nonconstant meromorphic function

proof: choose a divisor of degree $g+1$. $\square$

(3) $X$ has a nonzero meromorphic differential, and
$$\deg K_X = 2g-2$$

proof: let $f$ be a nonconstant meromorphic function and $f: X \longrightarrow \mathbb{P}^1$ the corresponding map.
If $\omega$ is any nonzero meromorphic diff on $\mathbb{P}^1$ then $F^*\omega$ is a nonzero meromorphic diff on $X$.
By Riemann-Hurwitz II we know that
$$\begin{aligned}\deg K_X &= \deg F^*K_{\mathbb{P}^1} + \deg R \\ &= (\deg F)\cdot(\deg K_{\mathbb{P}^1}) + \deg R \\ &= -2\cdot \deg F + \deg R\end{aligned}$$
By Riemann-Hurwitz I we know that
$$2g(X)-2 = -2\deg F + \deg R. \qquad \square$$

(4) Let $D$ be a divisor on $X$.

- If $\deg D \geq 2g-1$, then
$$h^0(X,D) = \deg D + 1 - g.$$
- If $\deg D \geq 2g$, then $D$ is base-point-free.
- If $\deg D \geq 2g+1$, then $D$ is very ample.

proof: • If $\deg D \geq 2g-1$, then $\deg K_X - D < 0$, so $h^0(X, \omega_X(-D)) = h^0(X, K_X - D) = 0$. The rest follows from Riemann-Roch.

• Recall that $D$ is base-point-free iff $h^0(X, D-p) = h^0(X, D) - 1$ for all $p \in X$. But this follows from the previous point.

• Recall that $D$ is very ample iff $h^0(X, D-p-q) = h^0(X, D) - 2$ for all $p, q \in X$. This follows again from the first point. $\square$

(5) Any compact Riemann surface is isomorphic to a smooth projective curve.

proof: Let $D$ be a divisor of degree $2g+1$. Then $D$ is very ample and it induces an embedding $\varphi : X \hookrightarrow \mathbb{P}^r$. $\square$

(6) $h^0(X, \omega_X) = g$.

proof: $h^0(X, K_X) = g - 1 + h^0(X, \mathcal{O}_X) = g$. $\square$