§7 : LINEAR SYSTEMS

X = compact Riemann surface

Rmk: Merom. functions on X are determined by their divisors, up to a nonzero constant. So we can study them via their divisors.

proof: Let $f_1, f_2 \in \mathbb{C}(X)$ merom. functions s.t. $\mathrm{div}(f_1) = \mathrm{div}(f_2)$. Then $\mathrm{div}\left(\frac{f_1}{f_2}\right) = 0$.
So $\frac{f_1}{f_2}$ has no poles, so it is holomorphic but since X is compact $\frac{f_1}{f_2} = \lambda \in \mathbb{C}^*$. $\square$

Some terminology:

A divisor $D = \sum n_p \cdot p$ on X is called EFFECTIVE if $n_p \geq 0$, and we write $D \geq 0$.

For two divisors D_1, D_2 we write $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$, i.e., $D_1 - D_2$ is effective.

Def: $H^0(X, D)$

Let X be a Riemann surface and D a divisor:

$$H^0(X, D) = \{ f \in \mathbb{C}(X)^* \mid \mathrm{div}(f) \geq -D \} \cup \{0\}$$

So these sets tell us about meromorphic functions with bounded poles.

__Example:__ (1) $X = \mathbb{P}^1$, $D = d\cdot\infty$ $d \in \mathbb{Z}$

$$H^0(\mathbb{P}^1, d\cdot\infty) = \{ f \in \mathbb{C}(\mathbb{P}^1) \mid \operatorname{div}(f) \geq -d\cdot\infty \}$$

So elements here are merom. functions with poles only at ∞ and of order at most d (if $d>0$) and with zeroes of order at least d (if $d<0$).

- $d<0$: $H^0(\mathbb{P}^1, d\cdot\infty) = 0$.

  Because any function in there has no poles, so it is constant. But it has also zeroes, so it must be zero.

- $d \geq 0$: $H^0(\mathbb{P}^1, d\cdot\infty) = \mathbb{C}[z]_{\leq d} = \mathbb{C}[s,t]_d$ $\quad z = t/s$

  If $f \in H^0(\mathbb{P}^1, d\cdot\infty)$ then it has poles only at ∞. Since any meromorphic function on $\mathbb{P}^1$ is rational it must be that f is a polynomial. Since the order of the pole at ∞ is at most d, this is a polynomial of degree $\leq d$.

We consider some properties of these sets:

(1) $H^0(X,D)$ is a vector space (subspace of $\mathbb{C}(X)$).

__proof__: because ord_p is a discrete valuation. $\square$

(2) If $D_1 \sim D_2$ ($\sim$ = linear equivalence) then there is a "canonical" isomorphism $H^0(X, D_1) \cong H^0(X, D_2)$.

<u>proof</u>: let $g \in \mathbb{C}(X)^*$ s.t.

$$\operatorname{div}(g) = D_1 - D_2$$

Then

$$\begin{aligned} H^0(X, D_1) &\xrightarrow{\cdot g} H^0(X, D_2) \\ f &\longmapsto f \cdot g \end{aligned}$$

is an isomorphism. If $f \in H^0(X, D_1)$

$$\begin{aligned} \operatorname{div}(f \cdot g) &= \operatorname{div}(f) + \operatorname{div}(g) \\ &= \operatorname{div}(f) + D_1 - D_2 \\ &\geq -D_1 + D_1 - D_2 = -D_2 \end{aligned}$$

So the map is well defined and the inverse is $(\cdot \frac{1}{g})$. $\square$

(3) There is a canonical identification

$$\mathbb{P}(H^0(X, D)) \cong |D| \stackrel{\text{def}}{:=} \left\{ \begin{array}{c} \text{all effective divisors} \\ E \sim D \end{array} \right\}$$

<u>proof</u>: Let $f \in H^0(X, D)$, $f \neq 0$. We can define an effective divisor

$$\boxed{\operatorname{div}^D(f) := \operatorname{div}(f) + D} \geq 0$$

This is also lin. equiv to D

$$\operatorname{div}^D(f) - D = \operatorname{div}(f)$$

Moreover if $\lambda \in \mathbb{C}^*$, then $\mathrm{div}^D(f) = \mathrm{div}^D(\lambda \cdot f)$

So we have a map

$$\begin{aligned} \mathbb{P}(H^0(X,D)) &\longrightarrow |D| \\ [f] &\longmapsto \mathrm{div}^D(f) = \mathrm{div}(f) + D \end{aligned}$$

- INJECTIVITY: if $\mathrm{div}^D(f_1) = \mathrm{div}^D(f_2)$, then $\mathrm{div}(f_1) = \mathrm{div}(f_2)$ and then we know that $f_1 = \lambda \cdot f_2$, $\lambda \in \mathbb{C}^*$.

- SURJECTIVITY: E effective, $E \sim D$. Then $E - D = \mathrm{div}(f)$ and $f \in H^0(X,D)$ because E is effective. Then $E = \mathrm{div}^D(f)$.

<u>Example</u>: (1) On $\mathbb{P}^1$, if $d \geq 0$

$$|d \cdot \infty| = \left\{ \begin{array}{l} \text{all effective divisors of} \\ \text{degree } d \text{ on } \mathbb{P}^1 \end{array} \right\}$$

$$= \mathbb{P}(\mathbb{C}[z]_{\leq d}) = \mathbb{P}(\mathbb{C}[s,t]_d)$$

(2) Let $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, and $p \in E$.

$|p| = \{p\}$ [Exercise 5.3.(c)]

$H^0(E,p) = \{ f \mid \mathrm{div}(f) \geq -p \}$

$= \langle 1 \rangle$

(4) If D is effective then $1 \in H^0(X,D)$ and indeed $D \in |D|$.

(5) If $\deg D < 0$, then $H^0(X,D) = 0$.

<u>proof</u>: $\mathbb{P}(H^0(X,D)) = \left\{ \begin{array}{c} \text{effective divisors} \\ E \sim D \end{array} \right\}$

But if $E \sim D$ then $\deg(E) = \deg(D) < 0$ impossible if E is effective. $\square$

(6) If $\deg D = 0$;

$$H^0(X,D) \cong \begin{cases} \mathbb{C} & \text{if } D \sim 0 \\ 0 & \text{if } D \not\sim 0 \end{cases}$$

<u>proof</u>: $\mathbb{P}(H^0(X,D)) = |D|$

So if $E \sim D$ and E is effective then $\deg(E) = 0$, so it must be $E = 0$. So

$$|D| = \begin{cases} \{0\} & , \text{if } D \sim 0 \\ \emptyset & , \text{if } D \not\sim 0 \end{cases}$$

Otherwise if $D \sim 0$ then

$$H^0(X,D) \cong H^0(X,0) = \{ f \mid \operatorname{div}(f) \geq 0 \}$$
$$= \{ \text{hol. functions on } X \} = \mathbb{C}.$$

(7) If $\deg D = 1$ then

$$\dim H^0(X,D) = \begin{cases} 2 & \text{if } X \cong \mathbb{P}^1 \\ \leq 1 & \text{if } X \not\cong \mathbb{P}^1 \end{cases}$$

proof: If $X \cong \mathbb{P}^1$, we saw this before:

$$H^0(X,D) \cong H^0(X,\infty) = \mathbb{C}[z]_{\leq 1}$$

On the other hand, suppose $\dim H^0(X,D) \geq 2$. Then there is an effective divisor E lin. equiv. to D and $E = p$. So

$$\dim H^0(X,p) = \dim H^0(X,D) \geq 2.$$

We always have $1 \in H^0(X,p)$ so there must be a nonconstant $f \in H^0(X,p)$, which has exactly one pole. So $f : X \longrightarrow \mathbb{P}^1$ is an isomorphism. $\square$

(8) If $p \in X$, then $H^0(X, D-p) \subseteq H^0(X,D)$.

In general if E is effective $H^0(X, D-E) \subseteq H^0(X,D)$.

proof: $H^0(X,D) = \{ f \mid \operatorname{div}(f) \geq -D \}$

$H^0(X,D-p) = \{ f \mid \operatorname{div}(f) \geq -D + p \}$. $\square$

Let's try to understand $H^0(X, D-p) \subseteq H^0(X, D)$:

Suppose first that p is not in the support of D

$\left(D = \sum n_p \cdot p \quad \mathrm{Supp}(D) = \{ p \mid n_p \neq 0 \} \right)$

Then each $f \in H^0(X, D)$ is holomorphic around p and then

$$H^0(X, D-p) = \{ f \in H^0(X, D) \mid \mathrm{div}(f) \geq -D + p \}$$
$$= \{ f \in H^0(X, D) \mid f(p) = 0 \}$$

Suppose in general that p appears in $\mathrm{Supp}(D)$

$$D = n_p \cdot p + \text{sum of pts distinct from } p$$

then

$$H^0(X, D) = \{ f \mid \mathrm{div}(f) \geq -n_p \cdot p + \dots \}$$
$$H^0(X, D-p) = \{ f \mid \mathrm{div}(f) \geq -(n_p - 1) p + \dots \}$$

Fix a local coordinate z around p, then if $f \in H^0(X, D)$ we can write a Laurent series expansion

$$f = a_{-n_p} z^{-n_p} + a_{-(n_p-1)} z^{-(n_p-1)} + \dots\dots$$

$$H^0(X, D-p) = \{ f \in H^0(X, D) \mid a_{-n_p} = 0 \}$$

(9) $H^0(X, D-p)$ has codimension ≤ 1 inside $H^0(X,D)$.

(10) In particular $h^0(X,D) \overset{def}{:=} \dim H^0(X,D)$ we see that this is finite, and moreover

$$h^0(X,D) = \deg D + 1 \quad , \text{ if } X \cong \mathbb{P}^1 \text{ and } \deg D \geq 0$$
$$h^0(X,D) \leq \deg D \quad , \text{ if } X \not\cong \mathbb{P}^1.$$

<u>proof</u>: let $d = \deg D$. If $X \cong \mathbb{P}^1$, then

$$H^0(X,D) \cong H^0(X, d \cdot \infty) \cong \mathbb{C}[z]_{\leq d}$$

and it has dim $d+1$ if $d \geq 0$ and 0 otherwise. Suppose $X \not\cong \mathbb{P}^1$. Then we have a chain of inclusions

$$H^0(X, D-(d-1)p) \subseteq \; \ldots\ldots \; \subseteq H^0(X, D-p) \subseteq H^0(X,D)$$

We know that

$$h^0(X, D-(d-1)p) \leq 1$$

because $\deg(D-(d-1)p) = 1$ and $X \not\cong \mathbb{P}^1$ and at each inclusion in the chain the dimension grows at most by 1, so

$$h^0(X,D) \leq h^0(X, D-(d-1)p) + \#\{\text{inclusions}\}$$
$$\leq 1 + d - 1 = d. \qquad \square$$

Def: LINEAR SYSTEM (of DIVISORS)

The proj space $|D|$ is called the complete linear system associated to D. A linear system is a linear subspace $\Delta \subseteq |D|$.
The degree of D is the degree of the linear system.
The dimension $r = \dim \Delta$ is the dimension of the linear system