MEROMORPHIC FUNCTIONS ON COMPLEX TORI

$\mathbb{P}^1$: meromorphic functions are rational: $\frac{p}{q}$ homogeneous of the same deg.

$$\mathbb{P}^1 = \mathbb{C}^2/\mathbb{C}^* \quad : \quad \frac{p(\lambda x)}{q(\lambda x)} = \frac{\lambda^d p(x)}{\lambda^d q(x)} = \frac{p(x)}{q(x)}$$

Complex torus: $\mathbb{C}/L$, where $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\mathbb{R}$-linearly independent $\omega_1, \omega_2 \in \mathbb{C}$

① Let f be an L-periodic holomorphic function, i.e. $f(x+z) = f(x)$ for all $z \in L$.
Then f induces a holomorphic function $\bar{f}: \mathbb{C}/L \to \mathbb{C}$
$\overset{\mathbb{C}/L \text{ compact}}{\Longrightarrow}$ $\bar{f}$ is constant

② But every meromorphic function on $\mathbb{C}/L$ is a quotient of theta functions

Without loss of generality, we can assume $L = \mathbb{Z} + \mathbb{Z}\tau$ with $\mathrm{Im}(\tau) > 0$

Def: (JACOBI) THETA FUNCTIONS

The theta function of $L = \mathbb{Z} + \mathbb{Z}\tau$ is

$$\theta_\tau(z) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i \left(n^2\tau + 2nz\right)\right)$$

Claim 1: This series converges absolutely and uniformly on compact subsets of $\mathbb{C} \times \underbrace{\mathbb{H}}_{=\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}}$

Proof: Suppose $|\mathrm{Im}(z)| < c \in \mathbb{R}$ and $\mathrm{Im}(\tau) > \varepsilon \in \mathbb{R}$

$$|\exp(\pi i\,(n^2\tau + 2nz))| = |\exp(-\pi\,(n^2\,\mathrm{Im}(\tau) + 2n\,\mathrm{Im}(z)))|$$
$$< \exp(-\pi\varepsilon n^2)\cdot\exp(2\pi c\cdot n) = \exp(-\pi\varepsilon)^{n^2}\cdot\exp(2\pi c)^n$$
$$= \exp(-\pi\varepsilon)^{n(n-n_0)}\cdot\left[\exp(-\pi\varepsilon)^{n_0}\cdot\exp(2\pi c)\right]^n$$

Choose $n_0 \in \mathbb{N}$ such that $\exp(-\pi\varepsilon)^{n_0}\cdot\exp(2\pi c) < 1$

so that $|\exp(\pi i\,(n^2\tau + 2nz))| < \exp(-\pi\varepsilon)^{n(n-n_0)}$

For large enough n, this is small enough.

series $\Rightarrow$ convergence

$$\sum_{n=-\infty}^{\infty}\exp(\pi i\,(n^2\tau + 2nz)) \text{ converges absolutely and uniformly on compact sets.}$$ □

So $\Theta_\tau(z) : \mathbb{C} \to \mathbb{C}$ is holomorphic.

<u>Claim 2:</u> $\Theta_\tau(z)$ is quasi-periodic with respect to $L = \mathbb{Z} + \mathbb{Z}\tau$

(a) $\Theta_\tau(z+1) = \Theta_\tau(z)$

(b) $\Theta_\tau(z+\tau) = \exp(-\pi i\tau - 2\pi i z)\,\Theta_\tau(z)$

<u>Proof:</u> (b) $\Theta_\tau(z+1) = \sum_{n=-\infty}^{\infty}\exp(\pi i\,(n^2\tau + 2n(z+\tau)))$

$$n^2\tau + 2n(z+\tau) = n^2\tau + 2n\tau + 2nz = (n+1)^2\tau - \tau + 2(n+1)z - 2z$$

(a) $$\Theta_\tau(z+\tau) = \sum_{n=-\infty}^{\infty}\exp(\pi i\,(n^2\tau + 2n(z+1)))$$

$$= \sum_{n=-\infty}^{\infty}\underbrace{\exp(2n\pi i)}_{=1}\exp(\pi i\,(n^2\tau + 2nz)) = \Theta_\tau(z).$$ □

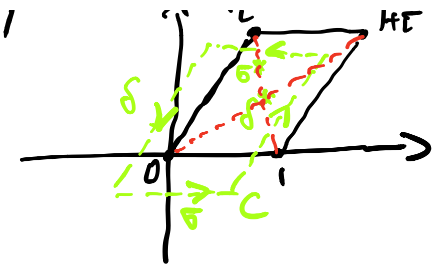
ZEROES OF THE THETA FUNCTION

<u>Step 1:</u> Count the zeroes in a parallelogram.

If f is holomorphic function, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \text{number of zeroes in the region bounded by } C$$

$$\log(\Theta(z+\tau))' = \log\left[\exp(-\pi i\tau - 2\pi i z)\,\Theta(z)\right]'$$
$$= -2\pi i + \left[\log(\Theta(z))\right]'$$



$$\oint_C \log(\Theta)'(z)\,dz$$

$$\int_\delta \frac{\Theta'}{\Theta}\,dz + \int_{\delta^*} \frac{\Theta'}{\Theta}\,dz = \int_\delta \frac{\Theta'(z)}{\Theta(z)}\,dz - \int_\delta \frac{\Theta'(z+1)}{\Theta(z+1)}\,dz = 0$$

$$\int_\sigma \frac{\Theta'(z)}{\Theta(z)}\,dz + \int_{\sigma^*} \frac{\Theta'(z)}{\Theta(z)}\,dz = \int_\sigma \frac{\Theta'(z)}{\Theta(z)}\,dz - \int_\sigma \frac{\Theta'(z+\tau)}{\Theta(z+\tau)}\,dz$$
$$= \int_\sigma \frac{\Theta'(z)}{\Theta(z)}\,dz - \int_\sigma -2\pi i\,dz - \int_\sigma \frac{\Theta'(z)}{\Theta(z)}\,dz$$
$$= 2\pi i$$

$\Rightarrow$ $\Theta_\tau(z)$ has one simple root in the fundamental parallelogram.

Step 2: Find the root.

$$\vartheta_{1/2,1/2}(-z,\tau) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i \left(n+\tfrac{1}{2}\right)^2\tau + 2\pi i\left(n+\tfrac{1}{2}\right)\left(-z+\tfrac{1}{2}\right)\right) =$$

$$\overset{m=-n-1}{=} \sum_{m=-\infty}^{\infty} \exp\left(\pi i\left(-m-\tfrac{1}{2}\right)^2\tau + 2\pi i\left(-m-\tfrac{1}{2}\right)\left(-z+\tfrac{1}{2}\right)\right) =$$

$$= \sum_{m=-\infty}^{\infty} \exp\left(\pi i\left(m+\tfrac{1}{2}\right)^2\tau + 2\pi i\left(m+\tfrac{1}{2}\right)\left(z+\tfrac{1}{2}\right) + 2\pi i\left(m+\tfrac{1}{2}\right)\right)$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{\exp\left(2\pi i\left(m+\tfrac{1}{2}\right)\right)}_{=-1} \cdot \exp\left(\pi i\left(m+\tfrac{1}{2}\right)^2\tau + 2\pi i\left(m+\tfrac{1}{2}\right)\left(z+\tfrac{1}{2}\right)\right)$$

$$= \sum_{m=-\infty}^{\infty} -\exp\left(\pi i\left(m+\tfrac{1}{2}\right)^2\tau + 2\pi i\left(m+\tfrac{1}{2}\right)\left(z+\tfrac{1}{2}\right)\right)$$

$$= -\vartheta_{\frac{1}{2},\frac{1}{2}}(z,\tau)$$

$z=0$: $\Rightarrow$ $\underbrace{\vartheta_{1/2,1/2}(0,\tau)}=0$ $\Big] \Rightarrow \Theta_\tau\left(\frac{1}{2}+\frac{1}{2}\tau\right)=0$

$$= \theta_\tau\left(\tfrac{1}{2}+\tfrac{1}{2}\tau\right)\cdot \exp\left(\pi i \tfrac{1}{4}\tau + 2\pi i \tfrac{1}{4}\right)$$

SUMMARY: $\theta_\tau(z)$ has one simple zero inside the fundamental parallelogram and that zero is $\frac{1}{2}+\frac{1}{2}\tau$.

So: $\theta_\tau(z) = 0 \iff z = \frac{1}{2}+\frac{1}{2}\tau + a + b\tau$ for some $a, b \in \mathbb{Z}$.

(and all those roots are simple)

<u>Def</u>: We write $\theta_\tau^{(x)}(z)$ for the theta function

$$\theta_\tau^{(x)}(z) = \theta\left(z - \tfrac{1}{2} - \tfrac{1}{2}\tau - x\right)$$

for any $x \in \mathbb{C}$.

The zeroes of $\theta_\tau^{(x)}(z)$ are $x + L$.

Exercise: $\theta_\tau^{(x)}(z+\tau) = -\exp(-2\pi i(z-x))\,\theta_\tau^{(x)}(z)$

Consider $R(z) = \dfrac{\prod_i \theta_\tau^{(x_i)}(z)}{\prod_j \theta_\tau^{(y_j)}(z)}$ $\to$ meromorphic function on $\mathbb{C}$

Is it L-periodic?

$R(z+1) = R(z)$: ✓

$$R(z+\tau) = \prod_{i=1}^{m} \theta_\tau^{(x_i)}(z+\tau) \Big/ \prod_{j=1}^{n} \theta_\tau^{(y_j)}(z+\tau) =$$

$$= \underbrace{\cancel{(-1)^{m-n}} \exp\left(\cancel{-2\pi i (m-n) z} + 2\pi i \left[\sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j\right]\right)}_{=1 \text{ for all } z \in \mathbb{C}} R(z)$$

$$\Rightarrow \boxed{m = n,\ \sum_{i=1}^{m} x_i - \sum_{j=1}^{m} y_j \in \mathbb{Z}}$$

We have shown:

Prop: Fix an integer $d \in \mathbb{N}$ and complex numbers $x_1, \dots, x_d \in \mathbb{C}$ and $y_1, \dots, y_d \in \mathbb{C}$ such that $\sum x_i - \sum y_j \in \mathbb{Z}$.
Then the ratio

$$R(z) = \prod_i \theta_\tau^{(x_i)}(z) \Big/ \prod_j \theta_\tau^{(y_j)}(z)$$

of translated theta functions is L-periodic & therefore induces a meromorphic function on $\mathbb{C}/L$.
The divisor of this function is $\sum 1\overline{x_i} - \sum 1\overline{y_j} \in \mathrm{Div}(\mathbb{C}/L)$

Prop: Any meromorphic function on a complex torus $\mathbb{C}/L$ is a ratio of translated theta functions.

Proof: Let f be a meromorphic function on $\mathbb{C}/L$. Then f has finitely many zeroes and poles, say $p_1, \dots, p_n$; $q_1, \dots, q_n$.
First, we show that $\sum p_i = \sum q_j$ in $\mathbb{C}/L$ as a group:
Suppose not and choose $p_0, q_0 \in X = \mathbb{C}/L$ such that

$$\sum_{i=0}^{n} p_i = \sum_{j=0}^{n} q_j.$$

Set $R(z) = \prod_i \theta_\tau^{(x_i)}(z) \Big/ \prod_j \theta_\tau^{(y_j)}(z)$

for points $x_i, y_j \in \mathbb{C}$ with $\overline{x_i} = p_i$, $\overline{y_j} = q_j$ and

$$\sum x_i - \sum y_j \in \mathbb{Z}.$$

Then $g = R/f$ is a meromorphic function on X with exactly one zero (p_0) and one pole (q_0) ↯ because $g(X) = 1$.

With the same argument we get that $f = R$ because their divisors are equal. $\square$