§4 MEROMORPHIC FUNCTIONS

Recall: meromorphic function on an open set $U \subseteq \mathbb{C}$ is the quotient of two holomorphic functions

$$f = \frac{g}{h} \quad , h \neq 0$$

order of f at z_0 $\quad = (z-z_0)^m \underbrace{f_0(z)}_{}$ locally around $z_0 \in U$

homomorphic at z_0, $f_0(z_0) \neq 0$

$\rightsquigarrow$ $\mathbb{C}(U) = \{$meromorphic functions on $U\}$

$$\mathrm{ord}_{z_0} : \mathbb{C}(U) \longrightarrow \mathbb{Z} \cup \{\infty\}$$

is a <u>discrete valuation</u>

<u>DEF:</u> (MEROMORPHIC FUNCTIONS ON RS)
A meromorphic function on a Riemann surface X is a collection $f_i = \frac{g_i}{h_i}$ of meromorphic functions on each chart $U_i \subseteq X$ such that $f_i = f_j$ on $U_i \cap U_j$
(that means $h_i g_j = h_j g_i$)

$\rightsquigarrow$ $\mathbb{C}(X) = \{$meromorphic function on $X\}$

& for all $p \in X$: $\mathrm{ord}_p : \mathbb{C}(X) \to \mathbb{Z} \cup \{\infty\}$

<u>Exm:</u> $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(z)$

MEROMORPHIC FUNCTIONS & HOLOMORPHIC MAPS TO $\mathbb{P}^1$

Let X be a Riemann surface and let f be a meromorphic function on X
$f = g_i / h_i$ on $U_i \subseteq X$

Define $F : X \to \mathbb{P}^1$ by gluing

$$F : U_i \to \mathbb{P}^1, \quad F(z) = [1, g_i/h_i(z)] =$$
$$= [h_i/g_i(z), 1]$$
$$= [h_i(z), g_i(z)]$$

Well-defined: locally at $z_0 \in U_i$

$g_i(z) = (z - z_0)^e g_0(z)$ with $g_0(z_0) \neq 0$
$h_i(z) = (z - z_0)^f h_0(z)$ $\quad h_0(z_0) \neq 0$

Set $m = e - f$

$$F(z) = \left[1, (z-z_0)^m \frac{g_0(z)}{h_0(z)}\right] = \left[(z-z_0)^{-m}, \frac{g_0(z)}{h_0(z)}\right] =$$

$$= [h_0(z), (z-z_0)^m g_0(z)] = [(z-z_0)^{-m} h_0(z), g_0(z)]$$

A pole of f gets mapped to $[0,1]$
a zero of f gets mapped to $[1,0]$

FROM MAPS TO $\mathbb{P}^1$ TO MEROMORPHIC FUNCTIONS

Let X be a Riemann surface and
$F: X \to \mathbb{P}^1$ be a holomorphic map
Let $p \in X$ and let $\Delta \subseteq X$ be a neighborhood of p, $\Delta \cong$ open neighborhood of $0 \in \mathbb{C}$
$p \cong 0$

Let z be a local coordinate

(1) $F(p) \neq 0, \infty$: Then $F: \Delta \to \mathbb{C}$ holomorphic
$\rightsquigarrow$ meromorphic function $f = \frac{F}{1}$ on Δ

(2) $F(p) = 0$:
Then $f = \frac{F}{1} = z^m \frac{F_0(z)}{1}$ with

$$F_0(0) \neq 0 :$$
$$\Rightarrow \operatorname{ord}_p(f) = \operatorname{mult}_0(F)$$

(3) $F(p) = \infty$: $F: \Delta \to \mathbb{P}^1$, $U_1 = \{x_1 \neq 0\} \subseteq \mathbb{P}^1$

$$F(z) = z^m F_0(z), \quad F_0(0) \neq 0$$

On $\Delta \setminus \{0\}$: $\quad \Delta\setminus\{0\} \to \mathcal{U}_1 \setminus \{\infty\} \cong \mathcal{U}_0 \setminus \{0\}$

$$z \mapsto \underbrace{\frac{1}{z^m F_0(z)}}_{f(z)}$$

meromorphic function f

$$\mathrm{ord}_p(f) = -\,\mathrm{mult}_0(F)$$

These local representations of f glue to a meromorphic function on X.

FACT: For any Riemann surface X, there is a correspondence

$$\left\{ \begin{array}{c} \text{meromorphic functions} \\ \text{on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{holomorphic maps} \\ X \to \mathbb{P}^1 \text{ (that} \\ \text{are not identically} \\ \infty \end{array} \right\}$$

ZEROES OF f $\longleftrightarrow$ $F^{-1}([1,0])$

POLES OF f $\longleftrightarrow$ $F^{-1}([0,1])$

$$\mathrm{ord}_p(f) = \begin{cases} \mathrm{mult}_p(F) & \text{if } F(p) = [1,0] \\ -\mathrm{mult}_p(F) & \text{if } F(p) = [0,1] \end{cases}$$

Prop: Let X be a compact Riemann surface,

be

Let $f \in \mathbb{C}(X)^{\times}$ non constant

$$\sum_{p \in X} \mathrm{ord}_p(f) = 0$$

<u>Proof:</u> $\displaystyle\sum_{p \in X} \mathrm{ord}_p(f) = \sum_{\substack{p \in X \\ \mathrm{ord}_p(f) > 0}} \mathrm{ord}_p(f) - \sum_{\substack{p \in X \\ \mathrm{ord}_p(f) < 0}} |\mathrm{ord}_p(f)|$

$$= \sum_{p \in F^{-1}([1,0])} \mathrm{mult}_p(F) - \sum_{p \in F^{-1}([0,1])} \mathrm{mult}_p(F) =$$

$$= \deg(F) - \deg(F) = 0 \qquad \square$$

<u>Lemma:</u> Let X be a compact Riemann surface and let f be a nonconstant meromorphic function on f
Then f has at least 1 pole.

<u>Proof:</u> $F: X \to \mathbb{P}^1$ holomorphic, non-constant

$\Rightarrow F$ is surjective. $\square$

<u>Lemma:</u> Let X be a compact Riemann surface with a meromorphic function with exactly one pole (counted with multiplicity).
Then $X \cong \mathbb{P}^1$

EXAMPLES: RATIONAL FUNCTIONS ON $\mathbb{P}^1$

→ Functions with arbitrary zeroes and poles

$f = \frac{(z-a)}{1}$ simple root at a and a simple pole at ∞

$f = \frac{1}{(z-a)}$ simple pole at a and a simple root at ∞

§5 DIVISORS

DEF (DIVISORS):

Let X be a compact Riemann surface

$\mathrm{Div}(X)$: free abelian group on X

$$\parallel$$

$$\left\{ \sum_{p\in X} n_p \cdot p \;\middle|\; n_p \in \mathbb{Z},\ n_p = 0 \text{ for almost all } p \in X \right\}$$

An element of $\mathrm{Div}(X)$ is called DIVISOR on X.

The DEGREE of a divisor $D = \sum_{p\in X} n_p \cdot p \in \mathrm{Div}(X)$ is $\deg(D) = \sum_{p\in X} n_p \in \mathbb{Z}$.

The map $\deg: \mathrm{Div}(X) \to \mathbb{Z}$ is a group homomorphism.

Let f be a meromorphic function on X

Define $\mathrm{div}(f) = \sum_{p \in X} \mathrm{ord}_p(f) \cdot p$

Every divisor of this form is called a

PRINCIPAL DIVISOR.

The set $\mathrm{PDIV}(X)$ of principal divisors is a subgroup of $\mathrm{DIV}(X)$

DIVISOR of ZEROES: $\mathrm{div}_0(f) = \sum_{\substack{p \in X \\ \mathrm{ord}_p(f) > 0}} \mathrm{ord}_p(f) \cdot p$

DIVISOR of POLES: $\mathrm{div}_\infty(f) = - \sum_{\substack{p \in X \\ \mathrm{ord}_p(f) < 0}} \mathrm{ord}_p(f) \cdot p$

$\leadsto \mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f)$

<u>Lemma:</u> Let f, g be meromorphic functions on X

ⓐ $\mathrm{div}(f \cdot g) = \mathrm{div}(f) + \mathrm{div}(g)$

ⓑ $\mathrm{div}(1/f) = -\mathrm{div}(f)$

(c) $\operatorname{div}\left(\frac{f}{g}\right) = \operatorname{div}(f) - \operatorname{div}(g)$

Proof: ord_p is a discrete valuation.

LINEAR EQUIVALENCE

Def:
Two divisors $D_1, D_2 \in \operatorname{DIV}(X)$ are LINEARLY EQUIVALENT, written $D_1 \sim D_2$, if their difference is a principal divisor i.e.

$$D_1 - D_2 = \operatorname{div}(f) \text{ for some } f \in \mathbb{C}(X).$$

Exm: linear equivalence on $\mathbb{P}^1$:

Every divisor of degree 0 on $\mathbb{P}^1$ is principal:

$$D = \sum_{\lambda_i \in \mathbb{C}} n_{\lambda_i} \lambda_i + e_\infty \cdot \infty$$

$$f = \prod_{n_{\lambda_i} \neq 0} (z - \lambda_i)^{n_{\lambda_i}} \Rightarrow \operatorname{div}(f) = D.$$

$$\Rightarrow \operatorname{Div}(\mathbb{P}^1) / \operatorname{PDiv}(\mathbb{P}^1) \cong \mathbb{Z}$$

Lemma: Let X be a compact Riemann surface

ⓐ Linear equivalence is an equivalence relation on $\mathrm{Div}(X)$

ⓑ $D \sim 0 \iff D \in \mathrm{PDiv}(X)$

ⓒ $D_1 \sim D_2 \Rightarrow \deg(D_1) = \deg(D_2)$

RAMIFICATION AND BRANCH DIVISORS

Let $F: X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces

The RAMIFICATION DIVISOR of F, denoted R_F, is

$$R_F = \sum_{p \in X} \left(\mathrm{mult}_p(F) - 1\right) \cdot p \in \mathrm{DIV}(X)$$

The BRANCH DIVISOR of F, denoted B_F, is

$$B_F = \sum_{q \in Y} \left(\sum_{p \in F^{-1}(q)} \left(\mathrm{mult}_p(F) - 1\right) \right) \cdot q \in \mathrm{DIV}(Y)$$

Hurwitz formula:

$$2g(Y)-2 = \deg(F)\,(2g(X)-2) + \deg(R_F)$$

INTERSECTION DIVISORS

Let X be a smooth projective curve (in $\mathbb{P}^2$)
Let $G(x,y,z) \neq 0$ be a homogeneous polynomial of degree d.

We define the INTERSECTION DIVISOR $\operatorname{div}(G)$ on X

Fix a point $p \in X$ where G vanishes and choose a polynomial H with $\deg(G) = \deg(H) = d$ which does not vanish at p
Then G/H is a meromorphic function on X

Define n_p to be the order of G/H at p

and $$\operatorname{div}(G) = \sum_{\substack{p \in X \\ G(p)=0}} n_p \cdot p$$

<u>Lemma:</u> well-defined...