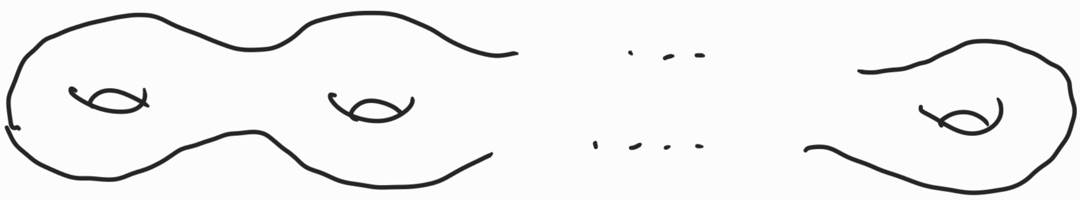
§3. TOPOLOGY of RIEMANN SURFACE

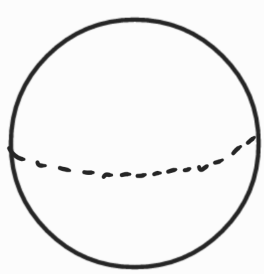
The topology of Riemann surfaces has been completly classified. Indeed, any compact Riemann surface is homeomorphic to a TORUS with g HOLES:



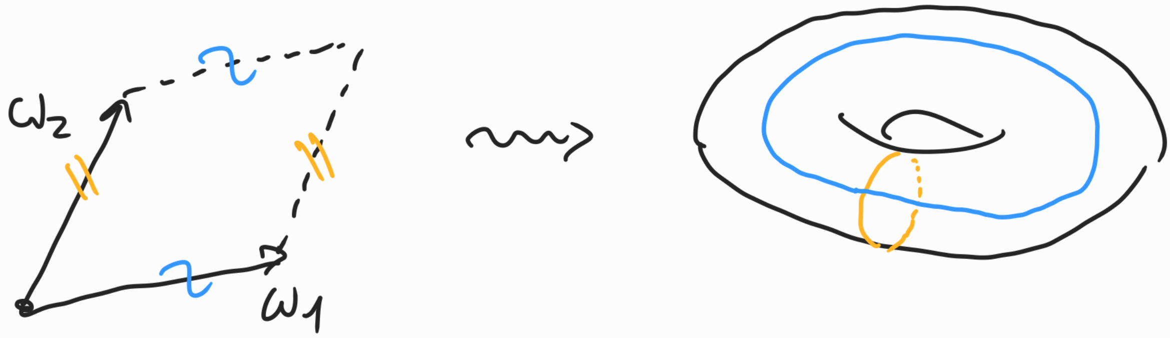
Def: TOPOLOGICAL GENUS
The (topological) genus of a compact Riemann surface X is the number g of holes

Examples: (1) $\mathbb{P}^1$: the projective line is homeomorphic to a sphere. This has zero holes so that the genus is 0.

$\mathbb{P}^1 \cong$



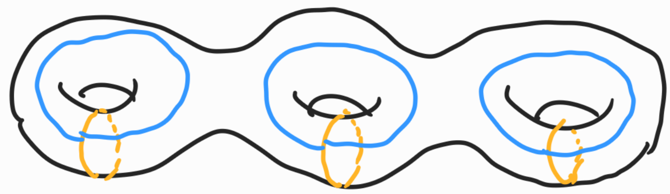
(2) A complex torus is a torus with one hole. Hence it has genus 1.



This number has other topological incarnations

- <u>HOMOLOGY</u>: The first homology group is free of rank $2g$:

$$H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$$



- <u>EULER CHARACTERISTIC</u>: The Euler characteristic is

$$\chi(X) = 2-2g.$$

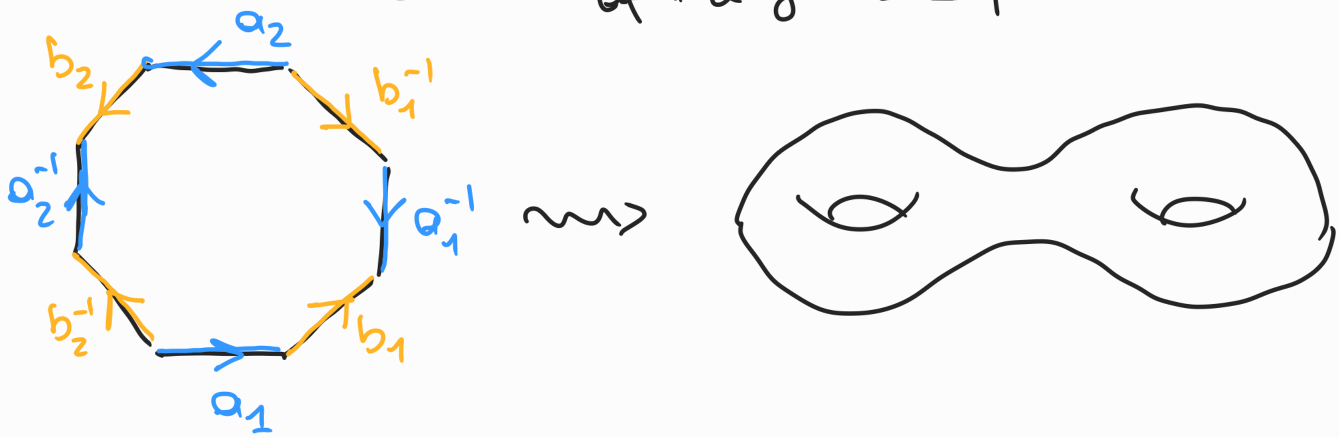
Indeed:

$$\chi(X) = \mathrm{rk}\, H^0(X,\mathbb{Z}) - \mathrm{rk}\, H^1(X,\mathbb{Z}) + \mathrm{rk}\, H^2(X,\mathbb{Z})$$
$$= 1 - 2g + 1 = 2-2g.$$

- <u>TOPOLOGICAL MODEL</u>: a Riemann surface of genus g can be obtained by taking a $4g$-gon with edges

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}$$

and identifying them. This generalizes the parallelogram of the genus 1 picture.



The genus, even if topological, is intimately related to the holomorphic structure of a Riemann surface. One of the most important examples is the Riemann-Hurwitz formula:

Thm: RIEMANN-HURWITZ FORMULA

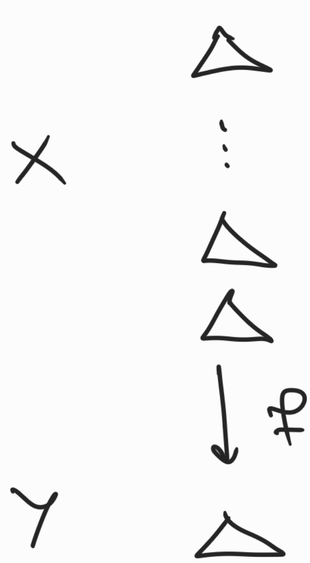
Let $f: X \to Y$ be a map of degree d between compact Riemann surfaces. Then

$$\boxed{2g(X)-2 = d \cdot (2g(Y)-2) + \sum_{p \in X} (\text{mult}_p(f) - 1).}$$

Rmk: Observe that $(\text{mult}_p(f)-1) > 0$ if and only if p is a ramification point for f. We know that these points are finitely many so that this sum makes sense.

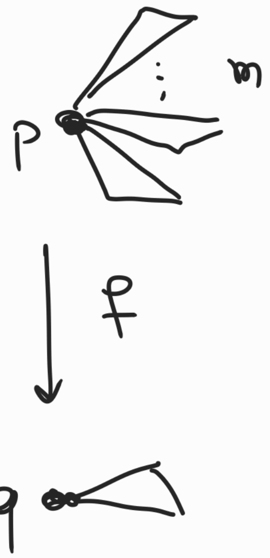
Proof: We will prove this in terms of Euler characteristics. Let's take a very fine triangulation on the Riemann surface Y, such that the branch points of $f: X \to Y$ are vertices.

The idea is to pull back this triangulation on X.



If a triangle on Y does not contain any branch point, then the preimage consists of d disjoint triangles.

Thus, the number of vertices, edges and faces is multiplied by d.



Suppose instead that a triangle has a vertex in a branch point $q \in Y$ and let $p \in f^{-1}(q)$ be a corresponding ramification point with $m = \text{mult}_p(f) > 1$.

Then the map locally looks like

$$z \longmapsto z^m.$$

Then the preimage consists of m triangles meeting at the point p.

In this case, the number of vertices, edges and faces is multiplied by d, except for the vertices over a branch point, whose number is

$$d - \sum_{p \in f^{-1}(q)} \left(\text{mult}_p(f) - 1 \right).$$

Hence, we see that

$$\begin{aligned}
\chi(X) &= |V_X| - |E_X| + |F_X| \\
&= d|V_Y| - \sum_{q \text{ branch}} \sum_{p \in f^{-1}(q)} \left(\text{mult}_p(f) - 1\right) - d|E_Y| + d|F_Y| \\
&= d\,\chi(Y) - \sum_{p \in X} \left(\text{mult}_p(f) - 1\right).
\end{aligned}$$

which is exactly the Riemann-Hurwitz formula. $\square$

As an application, let us compute the genus of a smooth plane curve:

Example: GENUS of a SMOOTH PLANE CURVE

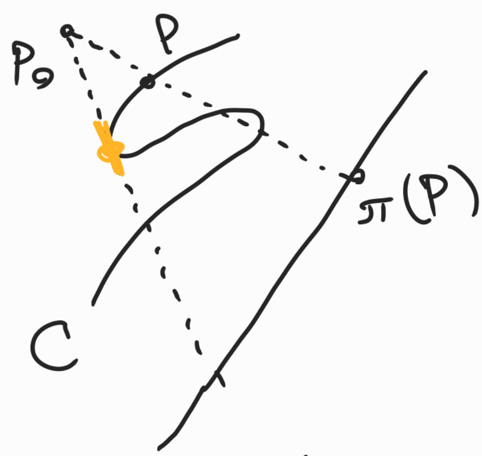
Let $C = \{F(X,Y,Z) = 0\}$ be a smooth plane curve of degree d, meaning that F is an homogeneous polynomial of degree d such that

$$\left\{ F = \frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Z} = 0 \right\} = \emptyset.$$

We claim that the genus of C is

$$\boxed{g(C) = \frac{(d-1)(d-2)}{2}}$$

Consider a general point $P_0 \in \mathbb{P}^2$ and let's consider the projection from P_0 onto a line $L \subseteq \mathbb{P}^2$



$$\pi : C \longrightarrow L \cong \mathbb{P}^1$$

What is the degree of this map?

If $Q \in L$ is a point, then

$$\pi^{-1}(Q) = \text{line}(P_0, Q) \cap C$$

So that $\pi^{-1}(Q)$ consists of d points, counted with multiplicity. Moreover we see that the ramification points are exactly those $P \in C$ such that $\text{line}(P_0, P)$ is tangent to C at P.

Now, one can show (in the Exercises :)) that, since P_0 is general, every line passing through P_0 is tangent to C with multiplicity at most two. In other words $\mathrm{mult}_P(\pi) \leq 2$, so that $\sum(\mathrm{mult}_P(\pi)-1)$ is given by the number of points $P \in C$ such that T_PC passes through P_0

Let's write $P_0 = [a_0, b_0, c_0]$. The tangent line to C at $P \in C$ is

$$T_PC = \left\{ \frac{\partial F}{\partial X}(P)X + \frac{\partial F}{\partial Y}(P)Y + \frac{\partial F}{\partial Z}(P)Z = 0 \right\}$$

Then this line passes through P_0 iff

$$\frac{\partial F}{\partial X}(P)a_0 + \frac{\partial F}{\partial Y}(P)b_0 + \frac{\partial F}{\partial Z}(P)c_0 = 0.$$

Hence,

$$\{P \in C \mid T_PC \ni P_0\} = \{F=0\} \cap \left\{ \frac{\partial F}{\partial X}\cdot a_0 + \frac{\partial F}{\partial Y}\cdot b_0 + \frac{\partial F}{\partial Z}c_0 = 0 \right\}$$

Since F has degree d and the derivatives degree $d-1$, BEZOUT THEOREM says that there are $d(d-1)$ such points. Now we use Riemann-Hurwitz:

$$\begin{aligned} 2g(C)-2 &= d\left(2g(L)-2\right) + \sum_{P \in C}\left(\mathrm{mult}_P(\pi)-1\right) \\ &= -2d + d(d-1), \end{aligned}$$

and then the formula for the genus follows. $\square$

§ 4: MEROMORPHIC FUNCTIONS

Recall that a meromorphic function on an open set $U \subseteq \mathbb{C}$ is a quotient of two holomorphic functions

$$f = \frac{g(z)}{h(z)} \qquad g, h \text{ holomorphic on } U, \; h \neq 0$$

<u>Rmk</u>: So, a meromorphic function is not a function, but a formal quotient of holomorphic functions. We will see later that they can be interpreted as maps $f : U \to \mathbb{P}^1$.

We can add and multiply meromorphic functions in the usual way; and with these operations, the set

$$\mathbb{C}(U) = \left\{ \begin{array}{c} \text{meromorphic} \\ \text{functions on } U \end{array} \right\} \quad \text{is a field.}$$

Now, let f be an holomorphic function and $z_0 \in U$. Then in a neighborhood of z_0 we can write

$$f = \frac{g(z)}{h(z)} = \frac{(z-z_0)^e g_0(z)}{(z-z_0)^f h_0(z)} \qquad \begin{array}{l} g_0, h_0 \text{ holomorphic} \\ g_0(z_0), h_0(z_0) \neq 0 \end{array}$$

$$= (z-z_0)^{e-f} \left(\frac{g_0(z)}{h_0(z)} \right)$$

$$= (z-z_0)^m f_0(z) \qquad \begin{array}{l} m \in \mathbb{Z} \\ f_0 \text{ holomorphic} \\ \text{and } f_0(z_0) \neq 0. \end{array}$$

<u>Def</u>: ORDER of a MEROMORPHIC FUNCTION

With the above notation, we define the order of the meromorphic function f at z_0 as

$$\mathrm{ord}_{z_0}(f) = m \quad \left[\mathrm{ord}_{z_0}(0) \overset{\mathrm{def}}{:=} \infty\right]$$

- $m > 0$: z_0 is called a ZERO of f of order m.
- $m < 0$: z_0 is called a POLE of f of order $-m$.

<u>Rmk</u>: The map

$$\mathrm{ord}_{z_0} : \mathbb{C}(U) \longrightarrow \mathbb{Z} \cup \{\infty\}$$

is a DISCRETE VALUATION. This means that

1) $\mathrm{ord}_{z_0}(f \cdot g) = \mathrm{ord}_{z_0}(f) + \mathrm{ord}_{z_0}(g)$.
2) $\mathrm{ord}_{z_0}(f + g) \geq \min\{\mathrm{ord}_{z_0}(f), \mathrm{ord}_{z_0}(g)\}$

as one can easily prove.

<u>Rmk</u>: Any holomorphic function $f(z)$ gives a meromorphic function $\frac{f(z)}{1}$ without poles. Conversely, any meromorphic function without poles comes from an holomorphic function.

We know that holomorphic functions have a local expansion into a power series. What about meromorphic functions?

Let f be meromorphic and $z_0 \in U$. Then we can write

$$f = (z-z_0)^m f_0(z) \qquad \begin{array}{l} f_0(z) \text{ holomorphic} \\ f_0(z_0) \neq 0 \end{array}$$

$$= (z-z_0)^m \sum_{n=0}^{\infty} a_{n,0} (z-z_0)^n$$

$$= \sum_{n=m}^{\infty} a_n (z-z_0)^n \qquad [a_n = a_{0,n-m}]$$

$$= a_m (z-z_0)^m + a_{m+1} (z-z_0)^{m+1} + \dots$$

This is a power series with finitely many terms with negative exponent (when $m<0$, i.e. when z_0 is a pole). Such a series is called a LAURENT SERIES.

All of this can be generalized to any Riemann surface

Def: MEROMORPHIC FUNCTION on a RIEMANN SURFACE
A meromorphic function on a Riemann surface S is a collection $f_i = \frac{g_i}{h_i}$ of meromorphic functions on each chart $U_i \subseteq S$. On $U_i \cap U_j$ we require that $f_i = f_j$ meaning $g_i h_j = g_j h_i$.

All the other notions extend to this general case:

$$\mathbb{C}(S) = \left\{ \begin{array}{c} \text{meromorphic functions} \\ \text{on } S \end{array} \right\} \quad \text{field}$$

$p \in S \quad \mathrm{ord}_p : \mathbb{C}(S) \to \mathbb{Z} \cup \{\infty\}$ discrete valuation.