§ 2 : RIEMANN SURFACE

A Riemann surface is a complex manifold of dimension one:

Def : RIEMANN SURFACE

A Riemann surface is a Hausdorff and second countable topological space S together with an open cover of charts $U_i \subseteq S$

$$\varphi_i : U_i \longrightarrow V_i \subseteq \mathbb{C} \quad \text{homeomorphisms}$$

such that the changes of coordinates are holomorphic:

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j).$$

Of course, nothing stops us from generalizing our previous definition of smooth manifolds to complex manifolds of arbitrary dimension, the only thing we need is the notion of a holomorphic function in multiple complex variables, but this is just a map

$$F : U \subseteq \mathbb{C}^n \longrightarrow \mathbb{C}^m$$
$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \longmapsto \begin{pmatrix} F_1(z_1, \dots, z_n) \\ \vdots \\ F_m(z_1, \dots, z_n) \end{pmatrix}$$

where each coordinate function F_i is holomorphic in each variable z_j.

Examples: (1) $\mathbb{P}^1(\mathbb{C})$ or the Riemann sphere.

This is the most basic Riemann surface. In general the calculations of before, show that $\mathbb{P}^n(\mathbb{C})$ is an n-dimensional complex manifold for any n. Hence, $\mathbb{P}^1(\mathbb{C})$ is a Riemann surface.

In particular, if $\mathbb{P}^1(\mathbb{C}) = \left\{ \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \right\}$ we have two charts

$U_0 = \{x_0 \neq 0\}$ with local coordinate $z = \frac{x_1}{x_0}$

$U_1 = \{x_1 \neq 0\}$ with local coordinate $w = \frac{x_0}{x_1} = \frac{1}{z}$

Since $\mathbb{P}^1(\mathbb{C}) = U_0 \cup \{[0,1]\} \cong \mathbb{A}^1_z \cup \{\infty\}$

the point $[0,1]$ is called point at infinity (w.r.t. the coordinate z).

(2) AFFINE PLANE CURVES

Let $f(z_1, z_2)$ be a nontrivial polynomial in two variables. We consider the affine curve

$$C = \{ f(z_1, z_2) = 0 \} \subseteq \mathbb{C}^2$$

A point $p \in C$ is regular if

$$\left(\frac{\partial f}{\partial z_1}(p), \frac{\partial f}{\partial z_2}(p) \right) \neq 0$$

Then around each regular point the affine curve C

is locally a Riemann surface. More precisely if

$\frac{\partial f}{\partial z_1}(p) \neq 0$, then z_2 is a local coordinate around p

$\frac{\partial f}{\partial z_2}(p) \neq 0$, then z_1 is a local coordinate around p

This works as in the case of smooth manifolds, via the implicit function theorem.

(3) PROJECTIVE PLANE CURVES

Let $F(X_0, X_1, X_2) \in \mathbb{C}[X_0, X_1, X_2]_d$ be an homogeneous polynomial of degree d, such that

$$\left\{ F = \frac{\partial F}{\partial X_0} = \frac{\partial F}{\partial X_1} = \frac{\partial F}{\partial X_2} \right\} \subseteq \mathbb{P}^2 \text{ is empty}$$

Then the zero locus $C = \{F = 0\} \subseteq \mathbb{P}^2$ is a Riemann surface. This can be c cked easily on the standard affine charts of $\mathbb{P}^2$.

Rmk: It turns out that the curve $C = \{F = 0\}$ is also connected, but we will prove this later.

(4) HYPERELLIPTIC CURVE

Let's consider a polynomial of even degree $2g+2 \geq 2$

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_{2g+2}) \qquad a_i \text{ distinct}$$

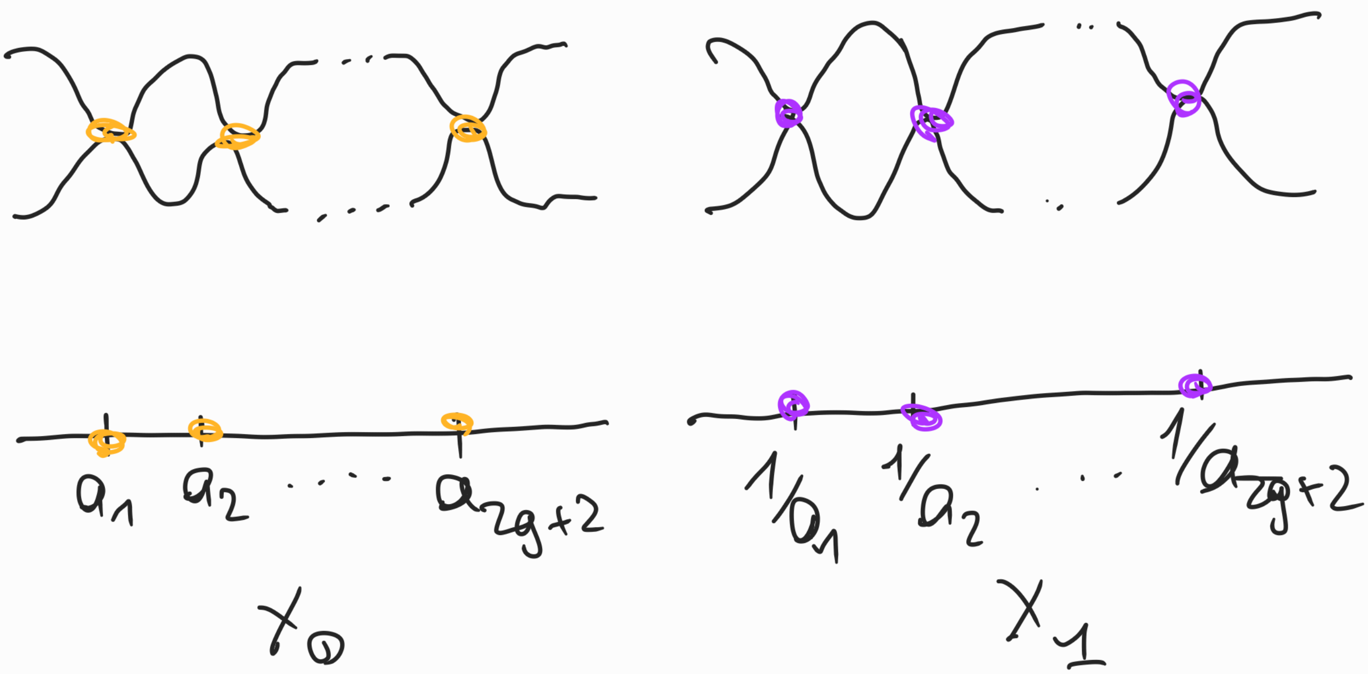
The hyperelliptic curve associated to this polynomial is obtained by gluing two open sets as follows:
First consider the polynomial

$$g(u) = u^{2g+2} \cdot f\left(\frac{1}{u}\right) = (1 - a_1 \cdot u) \cdot \ldots \cdot (1 - a_{2g+2} \cdot u)$$

then we have two affine curves with open subsets

$$X_0 = \{ y^2 = f(x) \} \supseteq U_0 = \{ y^2 = f(x),\ x \neq 0 \}$$

$$X_1 = \{ v^2 = g(u) \} \supseteq U_1 = \{ v^2 = g(u),\ u \neq 0 \}$$



Then we can glue X_0, X_1 along U_0, U_1 via the maps

$$\varphi : U_0 \longrightarrow U_1 \qquad \psi : U_1 \longrightarrow U_0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1/x \\ y/x^{g+1} \end{pmatrix} \qquad \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} 1/u \\ v/u^{g+1} \end{pmatrix}$$

Indeed, observe that if $y^2 = f(x)$ and $x \neq 0$, then

$$y^2 = f(x) \Longleftrightarrow y^2 = x^{2g+2} g(1/x) \Longleftrightarrow \left(\frac{y}{x^{g+1}}\right)^2 = g\left(1/x\right)$$

The resulting space $X = X_0 \cup X_1$ is a Riemann surface.

Moreover, the two natural maps

$$U_0 \to \mathbb{A}^1_x \quad (x,y) \mapsto x$$
$$U_1 \to \mathbb{A}^1_u \quad (u,v) \mapsto u$$

glue together to a global map

$$f : X \to \mathbb{P}^1.$$

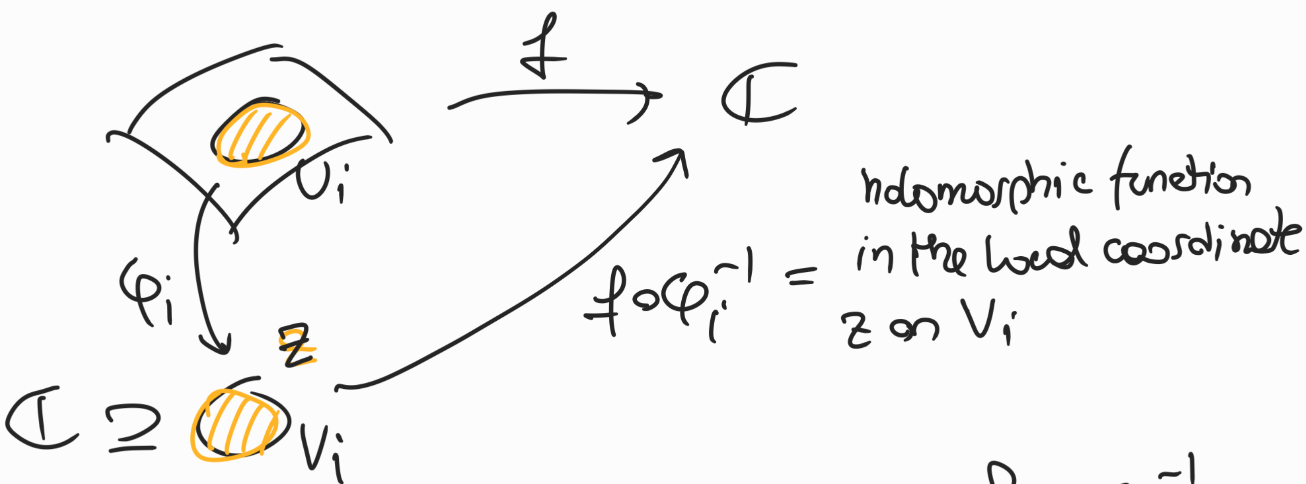
§ 2.1 : FUNCTIONS on RIEMANN SURFACES

Since on a Riemann surface we have local coordinates we can speak of holomorphic functions

Def : HOLOMORPHIC FUNCTION

Let S be a Riemann surface. A function $f : S \to \mathbb{C}$ is holomorphic if it is holomorphic in each local coordinate on S.

More formally, this means the following: we have an open cover of charts $S = \bigcup U_i$, $\varphi_i : U_i \xrightarrow{\sim} V_i \subseteq \mathbb{C}$



Then f is holomorphic if each composition $f|_{U_i} \circ \varphi_i^{-1}$ is holomorphic. This can be generalized to maps between any two Riemann surfaces

Def : HOLOMORPHIC MAP

A map $f : S_1 \to S_2$ between two Riemann surfaces is holomorphic if it is holomorphic in each local coordinate on S_1 and S_2.

Let $f : S_1 \to S_2$ be an holomorphic map between Riemann surfaces and let $p \in S_1$. Then Exercise 1.2 says that there are local coordinates around p and $f(p)$ such that f has the form

$$f(z) = z^e \qquad e > 0$$

Observe that e does not depend on the local coordinate. For example we can also define it as

$$e = \max \left\{ n \mid f^{(0)}(p) = f^{(1)}(p) = \dots = f^{(n-1)}(p) = 0 \right\}$$

Def: MULTIPLICITY of a MAP

With the above notation, we define the MULTIPLICITY of f at the point p as

$$\mathrm{mult}_p(f) = e.$$

Example: An holomorphic function $f : S \to \mathbb{C}$ has a zero at p if $f(p) = 0$ and $\mathrm{mult}_p(f) = e$.

Def: RAMIFICATION POINT

We say that an holomorphic map $f : S_1 \to S_2$ is ramified at p, if $\mathrm{mult}_p(f) > 1$.

§ 2.2 : MAPS OF COMPACT RIEMANN SURFACES

We will mostly care about COMPACT CONNECTED RIEMANN SURFACES, since these correspond to projective curves.

Holomorphic maps $f: S_1 \to S_2$ between two connected compact Riemann surfaces enjoy many nice properties:

(1) Any holomorphic map $f: S_1 \to S_2$ is either constant or surjective.

proof : if f is not constant then it is open (Exercise 1.3). Moreover f is closed because S_1 is compact. Hence $f(S_1)$ is both open and closed and as S_2 is connected it follows that $f(S_1) = S_2$

(2) Any nonconstant holomorphic map $f: S_1 \to S_2$ has finite fibers.

proof : from the local form of an holomorphic function, we can see that the fibers $f^{-1}(p)$ are discrete. Since S_1 is compact, $f^{-1}(p)$ must be compact and discrete, hence finite.

(3) Any nonconstant holomorphic map $f: S_1 \to S_2$ has finitely many ramification points.

proof : as for (2).

Now we can define the most important invariant of an holomorphic map between compact Riemann surfaces.

Prop/Def Let $f: S_1 \to S_2$ be a nonconstant holomorphic map between compact connected Riemann surfaces.

Then for any $q \in S_2$ the number of points in the fiber, counted with multiplicity, is constant:

$$\boxed{d(q) = \sum_{p \in f^{-1}(q)} \mathrm{mult}_p(f)}$$

This is called the DEGREE of the map.

proof: We will show that d is locally constant. Let $q \in S_2$ and let $f^{-1}(q) = \{p_1, \dots, p_n\}$. Then we can find charts $U_i \subseteq S_1$ around each of the p_i and $V \subseteq S_2$ around q such that in the corresponding local coordinates, f looks like $f(z) = z^{e_i}$ around each p_i.

Now, since S_1, S_2 are both compact, by shrinking V if needed, we can assume that $f^{-1}(V) \subseteq U_1 \cup \dots \cup U_n$ (exercise in set topology).

Now we show that the function is constant on V. Let $q' \in V$ then $f^{-1}(q') \subseteq U_1 \cup \dots \cup U_n$. Then we can simply count the points of the fiber in each U_i: on each of those the map looks like $f(z) = z^{e_i}$

and it is now clear that $f^{-1}(a) = \begin{cases} \text{one pt of mult } e_i & , \text{if } a = 0 \\ e_i \text{ distinct pts} & , \text{if } a \neq 0 \end{cases}$

In any case, the sum of pts with multiplicity is e_i. Hence $d(q') = \sum e_i \ \forall q' \in V$. $\square$