# RIEMANN SURFACES and ALGEBRAIC CURVES

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LECTURES: Wed, 15-17, Uni Leipzig Room P801

EXERCISES: Wed, 11-13, MPI-MiS, Room E1 05

ONLINE: If needed, the course might take place on Zoom.

WEBPAGE: https://personal-homepages.mis.mpg.de/agostini

REFERENCES:
- Notes on the webpage
- CAVALIERI and MILES, Riemann Surfaces and algebraic curves
- FULTON, Algebraic curves.
- KIRWAN, Complex Algebraic curves
- MIRANDA, Algebraic curves and Riemann surfaces

.... others on the webpage.

EXERCISES: • On the webpage.

## § 0 : MOTIVATION: ABELIAN INTEGRALS

Say we want to compute the integral

$$\int_a^b \frac{1}{(x-1)(x-2)(x-3)}\, dx .$$ How do we do it?

Well, we can compute a decomposition

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{1}{2}\frac{1}{(x-1)} - \frac{1}{(x-2)} + \frac{1}{2}\frac{1}{(x-3)}$$

and then it is easy to compute the primitive

$$\int \frac{1}{(x-1)}\, dx = \log(x-1), \quad \int \frac{1}{(x-2)}\, dx = \log(x-2), \ldots$$

The same strategy works for any integral of a RATIONAL FUNCTION $\frac{p(x)}{q(x)}$, where $p(x), q(x)$ are two polynomials.

We can always write

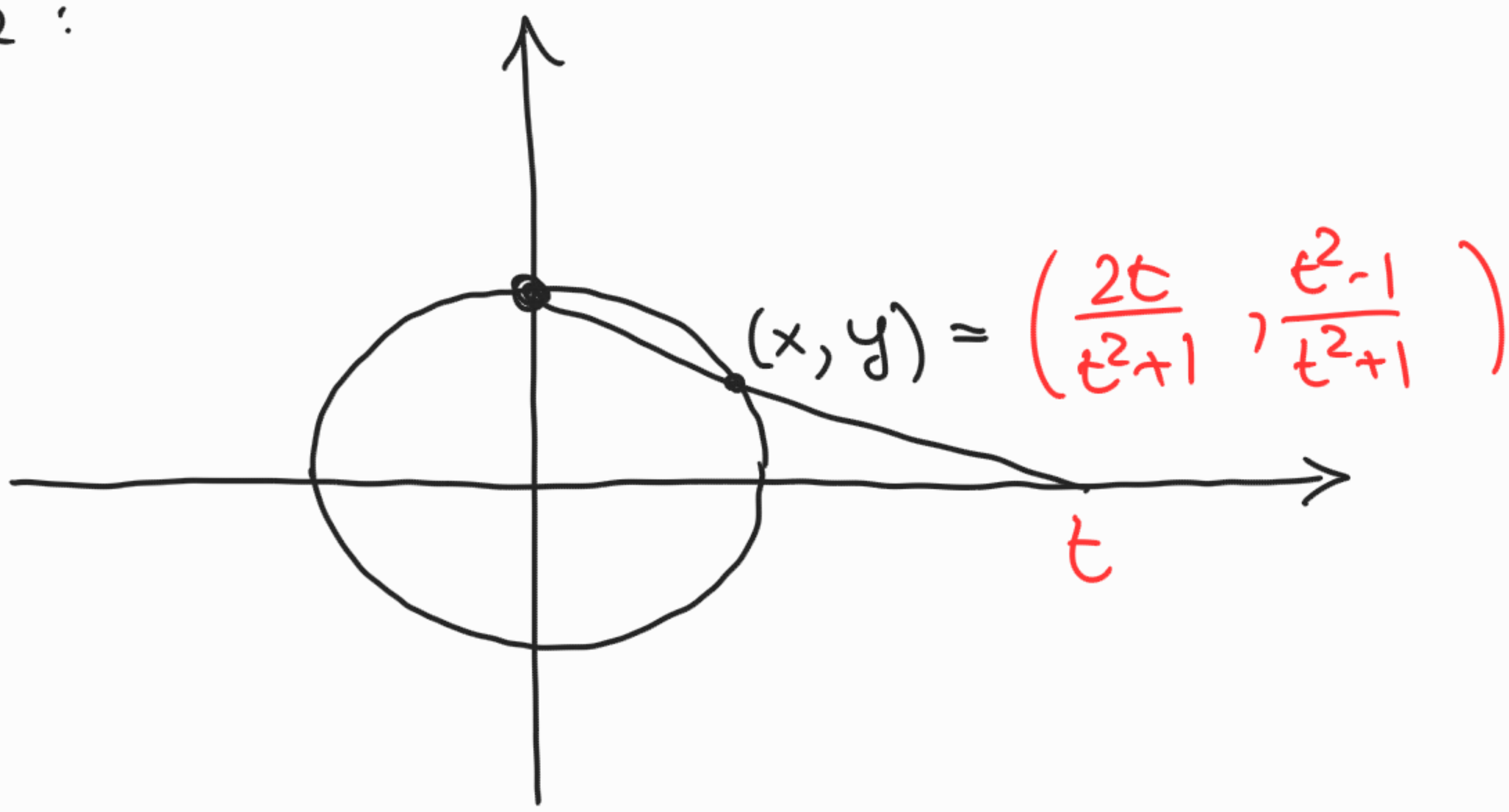
$$\frac{p(x)}{q(x)} = r(x) + a_1 (x-b_1)^{-k_1} + \ldots + a_m (x-b_m)^{-k_m}$$

where $r(x)$ is a polynomial, and we know how to integrate these.

Let's consider now the integral

$$\int_a^b \frac{1}{\sqrt{1-x^2}}\,dx$$

We set $y=\sqrt{1+x^2}$, so that $y^2+x^2=1$.
This is an algebraic curve in the plane, the familiar circle:




This curve can be parametrized by a rational parameter $t$ via projection from the point $(1,0)$:

$$x=\frac{2t}{t^2+1}\ , \quad y=\frac{t^2-1}{t^2+1}$$

Then the integral becomes

$$\int \frac{1}{y}\,dx = \int \frac{t^2+1}{t^2-1}\, d\left(\frac{2t}{t^2+1}\right) = \int \frac{\cancel{t^2+1}}{t^2-1}\ \frac{2(t^2+1)-4t^2}{(t^2+1)^{\cancel{2}}}\,dt$$

$$= \int \frac{-2t^2+2}{(t^2-1)(t^2+1)}\,dt = -2\int \frac{1}{t^2+1}\,dt$$

and this is again the integral of a rational function.

The same idea works when we try to compute integrals of the form

$$\int R(x, y)\, dx \quad \text{where } y = \sqrt{ax^2+bx+c}$$

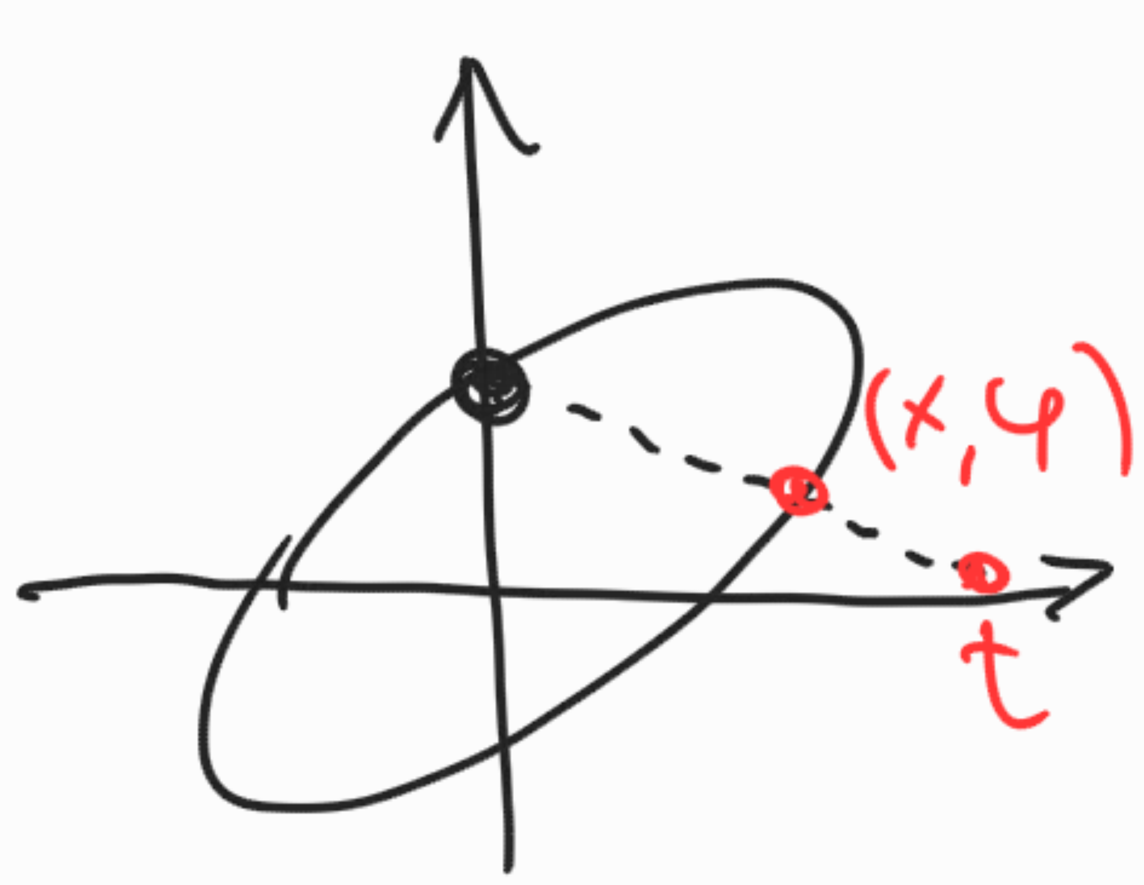
Why does this work? Well the idea is that all curves

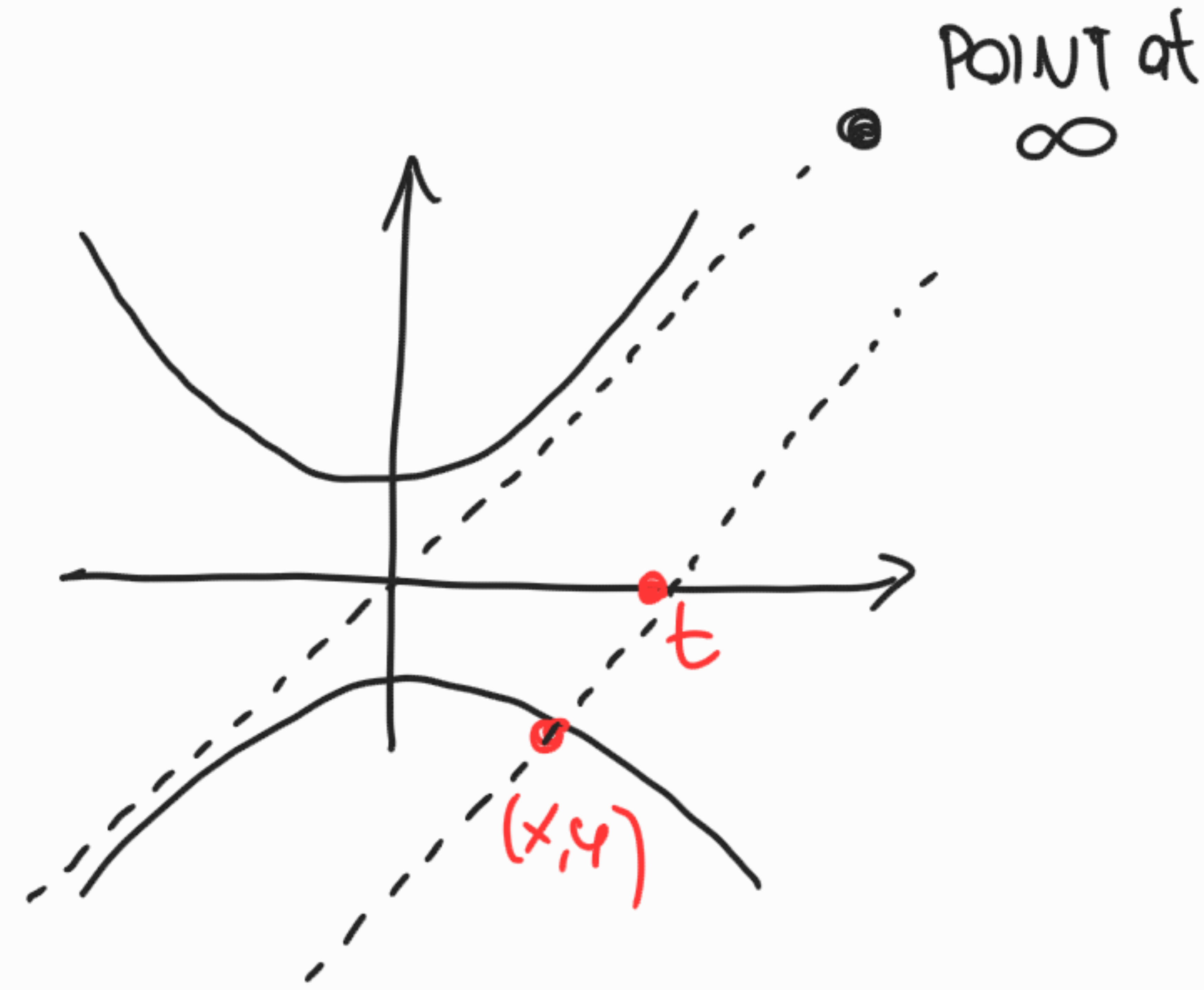
$$\{ y^2 = ax^2+bx+c \}$$

can be parametrized in the form

$x = x(t)$, $y = y(t)$ so that $x(t), y(t)$ are rational functions of $t$.

The parametrization is by projection from a point, just like before




This means that the algebraic curve

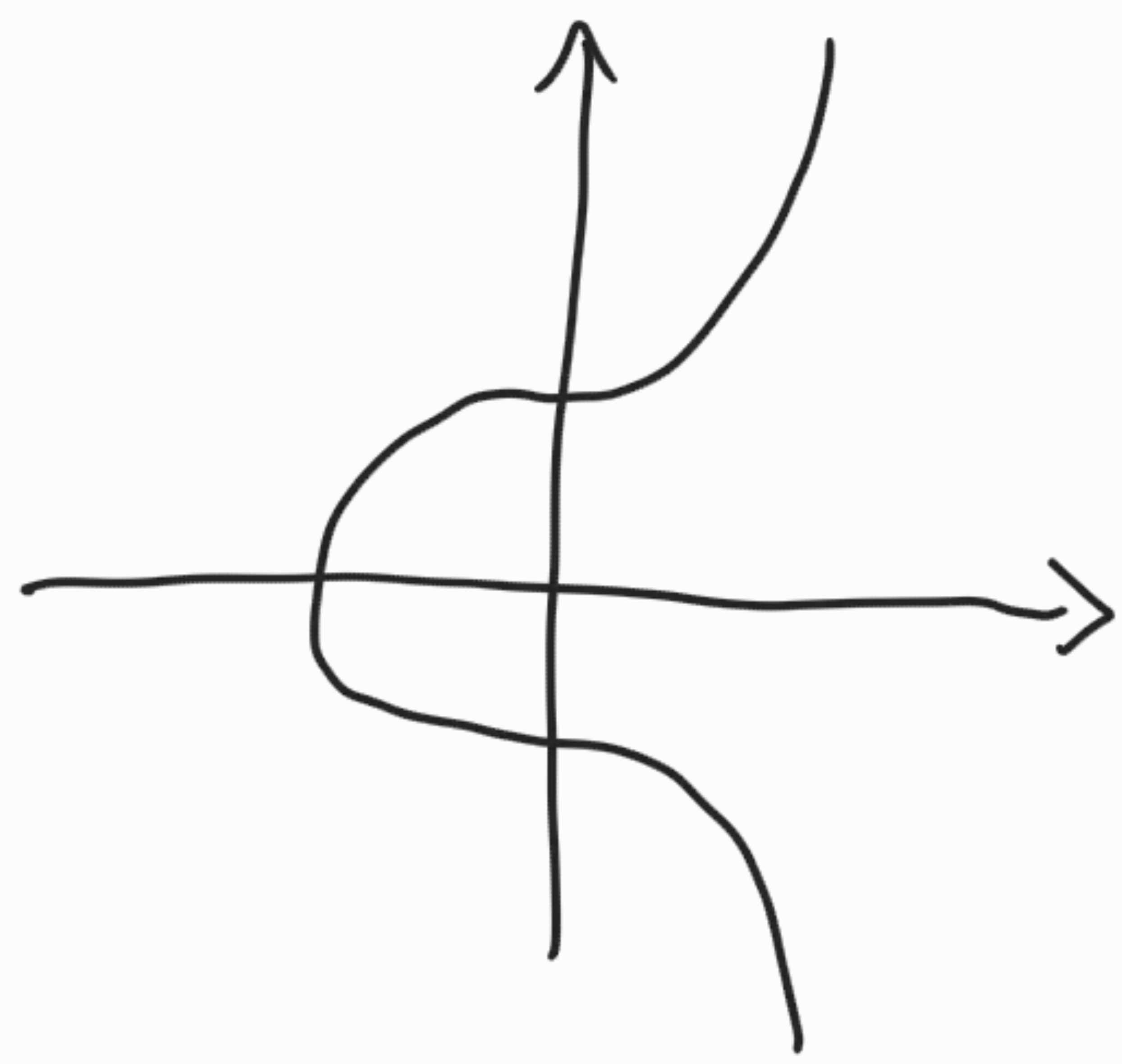
$C = \{ y^2 = ax^2+bx+c \}$ is RATIONAL, i.e. it can be parametrized by rational functions of $t$.

What about the integral

$$\int \frac{1}{\sqrt{x^3+x+1}}\, dx \quad ?$$

Can we do it in the same way? It turns out that the answer is <u>**NO**</u>! That's because the curve

$$C' = \{ y^2 = x^3+x+1 \}$$



is <u>not</u> rational. This is an example of elliptic curve.

Much of the theory of Riemann surfaces and algebraic curves arose from these integrals, and later in the course we are going to see how to compute them.

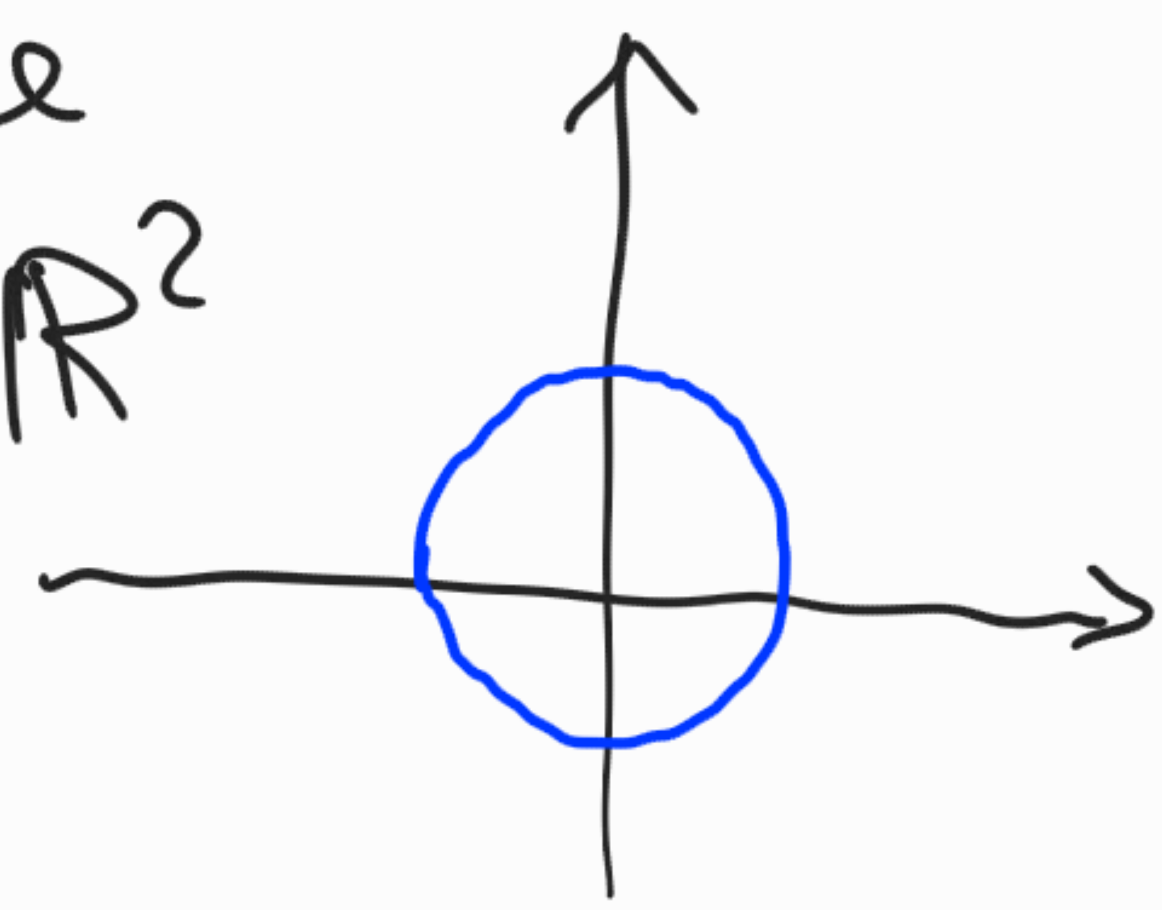
# §1 : COMPLEX MANIFOLDS and HOLOMORPHIC FUNCTIONS.

The concept of MANIFOLD is one of the fundamental insights of $\geq$ 19th century mathematics.

Before that, geometric objects existed mostly only extrinsecally, usually as subsets of the affine space $\mathbb{R}^n$:

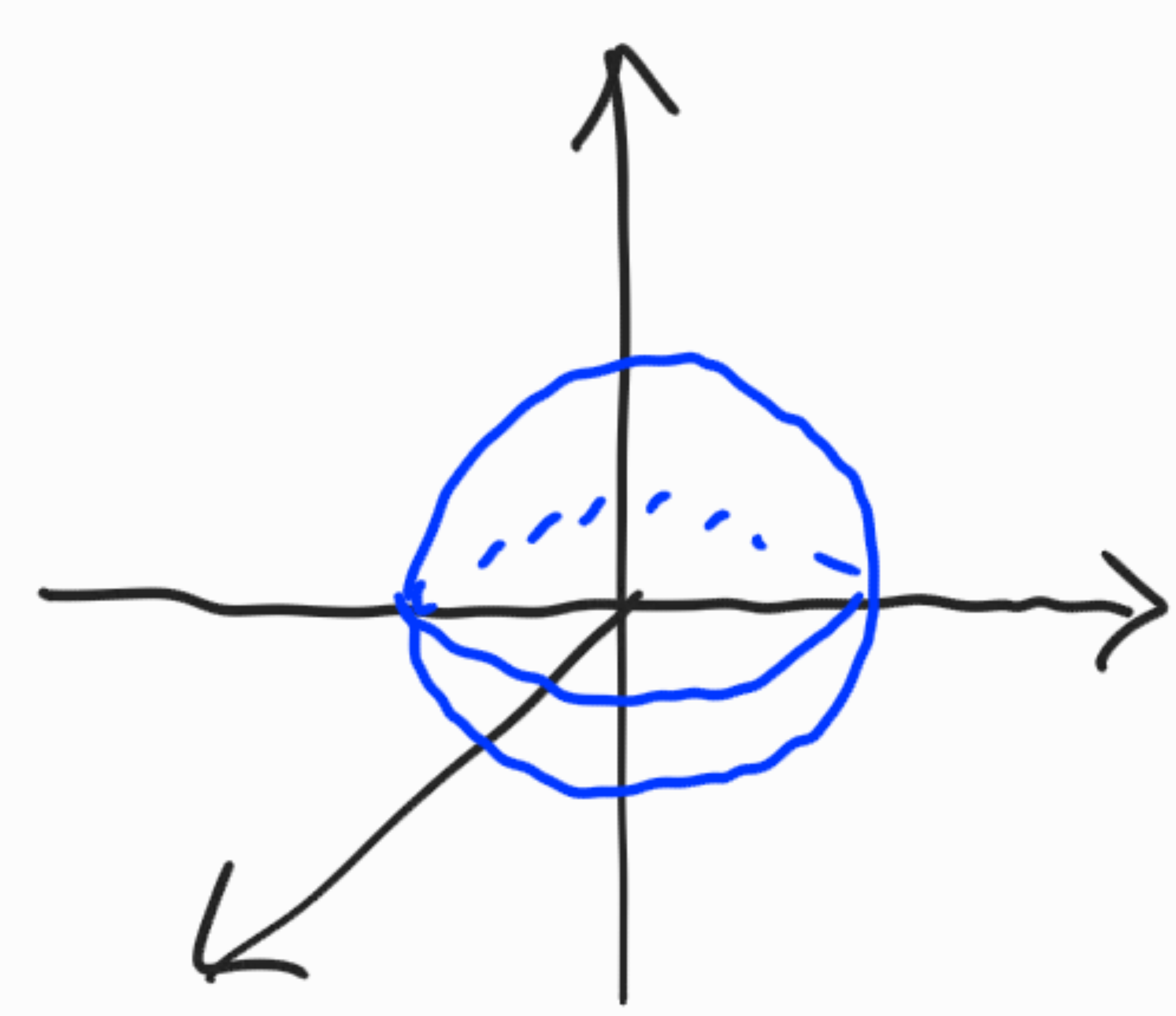
Examples: (1) The circle

$$\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}\subseteq\mathbb{R}^2$$



(2) The sphere

$$\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$$



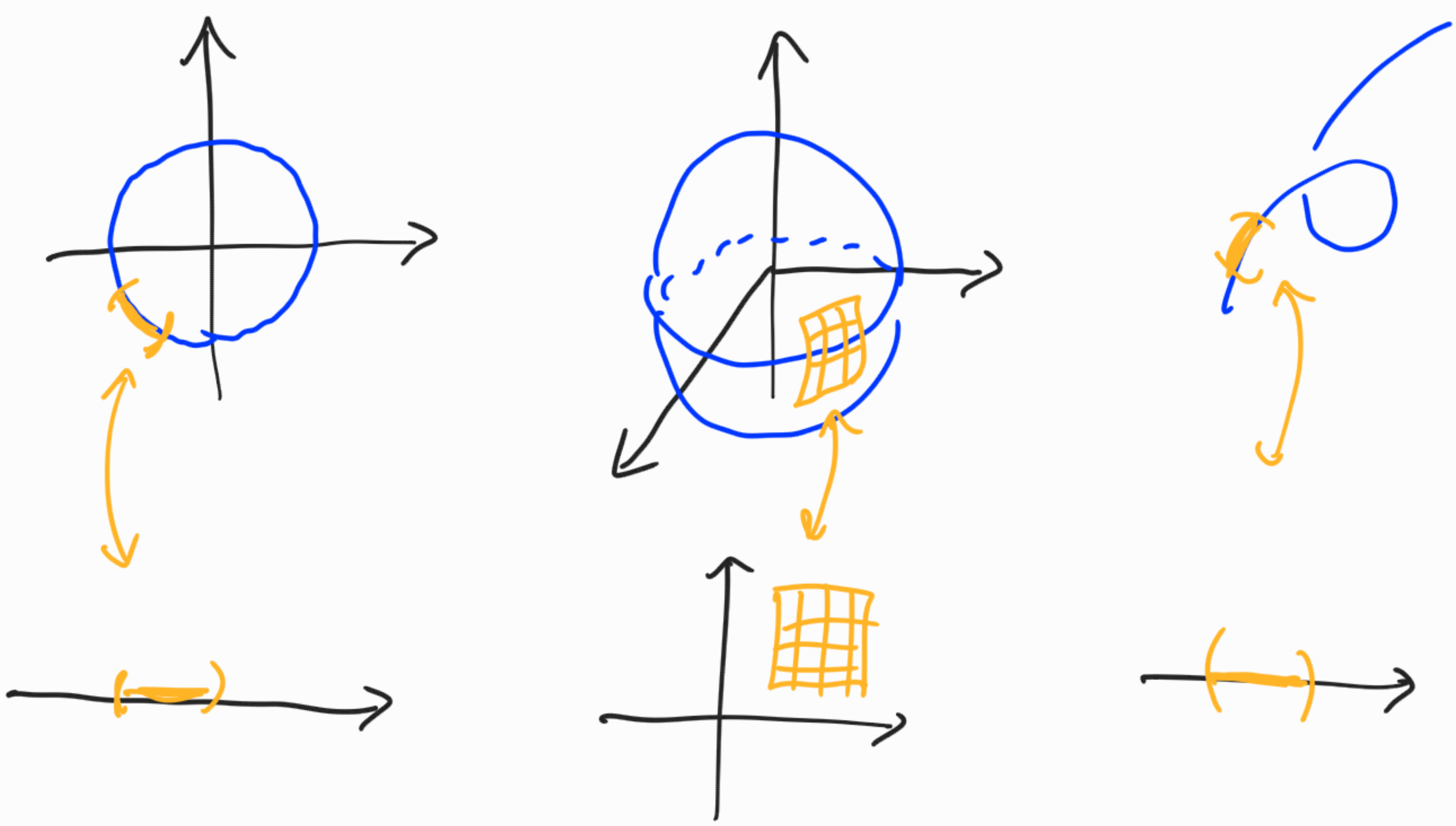
(3) A parametric curve

$$\{(t,t^2,t^3)\in\mathbb{R}^3 \mid t\in\mathbb{R}\}$$

However, RIEMANN realized that geometric objects can exist INTRINSICALLY, that is, without the need of an AMBIENT SPACE.

This leads to the concept of (SMOOTH) MANIFOLD:

SMOOTH MANIFOLD of dimension $n$ = a topological space that locally looks like the affine space $\mathbb{R}^n$.



Alternatively, this means that on a manifold of dimension $n$ we have local coordinates $(x_1, x_2, \dots, x_n)$ around each point.

In particular we can make CALCULUS (i.e. differentiate and integrate on such manifolds).

Let's now give the formal definition of a manifold.

**Def:** SMOOTH MANIFOLD

A smooth manifold is a second countable and Hausdorff topological space $X$ together with an open cover

$$X = \bigcup_{i \in I} U_i \quad \text{s.t.}$$

(i) there are homeomorphisms, called CHARTS

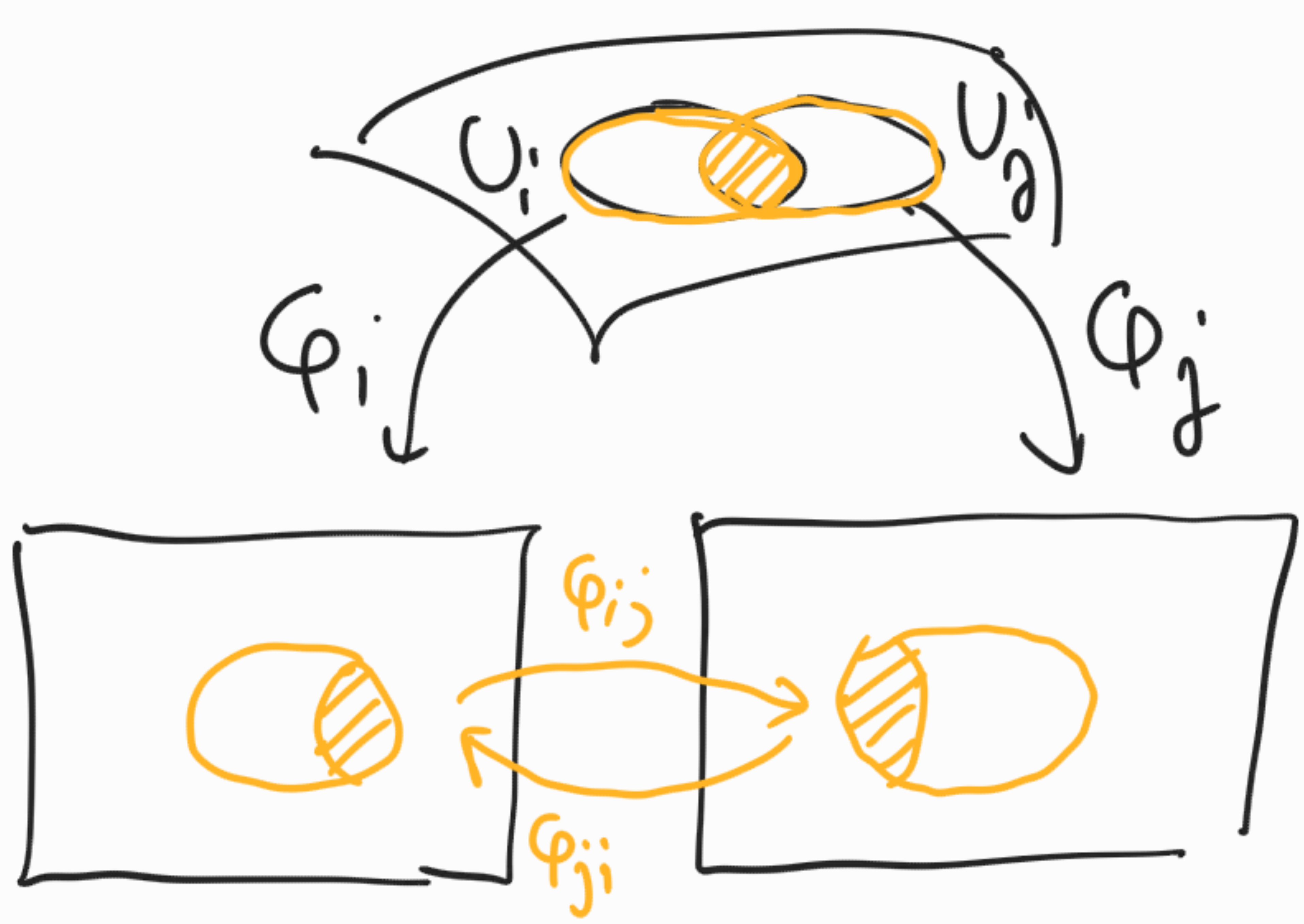
$$\varphi_i : U_i \xrightarrow{\sim} V_i$$

where $V_i \subseteq \mathbb{R}^n$ is an open subset

(ii) these CHARTS are compatible, meaning that the CHANGE of COORDINATES MAPS

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are SMOOTH $(= C^\infty)$



<u>Example</u>: PROJECTIVE SPACE $\mathbb{P}^n$

This is an extremely important example for us. For starters, it is a manifold that does not appear naturally as a subset of something else.

$\mathbb{P}^n_{\mathbb{R}} = \{ [x_0, \dots, x_n] \}$ space of $(n+1)$-tuples s.t.

$$\begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \neq 0, \quad \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix} \overset{\text{def}}{\iff} \exists \lambda \in \mathbb{R}^* \text{ s.t. } \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$$

Why is this a manifold? Consider the charts

$U_i = \left\{ \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \middle| \, x_i \neq 0 \right\}$. These have homeomorphisms

$$\varphi_i : U_i \xrightarrow{\sim} \mathbb{R}^n \qquad \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} x_0/x_i \\ \vdots \\ \widehat{x_i/x_i} \\ \vdots \\ x_n/x_i \end{bmatrix}$$

from $U_i$ to the affine space $\mathbb{R}^n$ with coordinates $(x_{0i}, x_{1i}, \dots, \hat{x}_{ii}, \dots, x_{ni})$

So, these charts put on each $U_i$ the coordinates

$$x_{ai} = \frac{X_a}{X_i}$$

Now, we should check that the changes of coordinates are smooth. Why should we expect this? Well, on the intersection

$$U_i \cap U_j = \{ x_i \neq 0, x_j \neq 0 \}$$

we have the two sets of coordinates

$$(x_{0i}, x_{1i}, \dots, x_{ni}) \text{ with } x_{ji} \neq 0$$

and $$(x_{0j}, x_{1j}, \dots, x_{nj}) \text{ with } x_{ij} \neq 0$$

How do we change from one to the other? We see that

$$x_{aj} = \frac{X_a}{X_j} = \frac{X_a}{X_i} \cdot \frac{X_i}{X_j} = \frac{\left(\frac{X_a}{X_i}\right)}{\left(\frac{X_j}{X_i}\right)} = \frac{x_{ai}}{x_{ji}}$$

Thus, to go from one set of coordinates to the other, we simply apply multiplication by $1/x_{ji}$ which is clearly a smooth map.

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In our course we will not be concerned with smooth manifolds but, rather, with COMPLEX MANIFOLDS. These are spaces that locally look like the complex affine space $\mathbb{C}^n$. Then, we will have to deal with functions that are differentiables in the complex sense: HOLOMORPHIC FUNCTIONS.

## § 1.2 : HOLOMORPHIC FUNCTIONS

Recall the definition

<u>Def</u>: HOLOMORPHIC FUNCTION
Let $U \subseteq \mathbb{C}$ be open. A function

$$f : U \longrightarrow \mathbb{C}$$

is holomorphic if at each point $z \in U$ the limit

$$f'(z) \overset{\text{def}}{:=} \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists, where $h$ is a complex number.

This is similar to the definition of differentiable functions. However the condition of being holomorphic is <u>much stronger</u>: indeed

$$\boxed{\begin{array}{l}\text{HOLOMORPHIC FUNCTIONS are ANALYTIC}\\ \text{Let } f : U \to \mathbb{C} \text{ be an holomorphic function. Then}\\ \text{for each } z_0 \in U \text{ there is a disk around } z_0\\ z_0 \in \Delta \subseteq U \text{ s.t.}\\ \qquad f(z) = \sum_{n \geq 0} a_n (z - z_0)^n \quad \forall z \in \Delta\\ \text{Conversely, any analytic function is holomorphic.}\end{array}}$$

In the Exercises, we will see some of the consequences
One is:

**Fact**: Holomorphic functions are determined on an open set:

Let $U \subseteq \mathbb{C}$ be an open connected subset and $f, g : U \to \mathbb{C}$ two holomorphic functions. Then TFAE:

(i) $f = g$ on $U$.

(ii) $f = g$ on a nonempty open subset of $U$.

(iii) $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all $k \geq 0$ and one $z_0 \in U$.

**Proof**: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear.

(iii) $\Rightarrow$ (i) Replacing $f$ with $f-g$ we can assume that $g = 0$. Then consider the set

$$\Omega = \{ z \in U \mid f^{(k)}(z) = 0 \ \forall k \geq 0 \}$$

If we show that this is open, closed and nonempty, then since $U$ is connected it must be that $\Omega = U$ and we are done.

- $\Omega$ is closed because $\Omega = \bigcap_{k \geq 0} \{ z \mid f^{(k)}(z) = 0 \}$ is an intersection of closed.
- $\Omega$ is nonempty because $z_0 \in \Omega$.
- $\Omega$ is open: suppose $w \in \Omega$. Then in a neighborhood of $w$ we can write

$$f(z) = \sum f^{(k)}(w)(z-w)^k = 0.$$

$\square$