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Riemann Surfaces and Algebraic Curves

WiSe 2020/21

Bonus Sheet 1

This will be (maybe) be discussed on November 25

Exercise (Dual plane curves) We consider the projective plane $\mathbb{P}^2$ with homogeneous coordinate $[X, Y, Z]$. The *dual projective plane* $\mathbb{P}^{2*}$ is another projective plane with homogeneous coordinates $[a, b, c]$ that parametrizes lines in $\mathbb{P}^2$. More precisely, the point $[a, b, c] \in \mathbb{P}^{2*}$ corresponds to the line

$$\{aX + bY + cZ = 0\} \subseteq \mathbb{P}^2.$$

a) Show that there is a natural identification $(\mathbb{P}^{2*})^* \cong \mathbb{P}^2$.

Let $C \subseteq \mathbb{P}^2$ be a possibly singular plane curve defined by an irreducible homogeneous polynomial $F(X, Y, Z)$ of degree d. We denote by $C_{reg} \subseteq C$ the open subset of regular points, so that for each $p \in \mathbb{C}_{reg}$ there is a well-defined tangent line $\mathbb{T}_pC$ to C at p. The *Gauss map* is the map

$$\gamma \colon C_{reg} \longrightarrow \mathbb{P}^{2*}, \qquad p \mapsto \mathbb{T}_pC$$

b) Show that γ is defined by homogeneous polynomials and show that it is non constant as soon as $d \geq 2$.

The closure of $\gamma(C_{reg})$ inside $\mathbb{P}^2$ is a plane curve $C^\vee$, called the *dual curve* of C. Let's make this concrete with a couple of examples:

c) Give an equation for the dual curves of the three curves $C_1 = \{X^2 + Y^2 + Z^2 = 0\}, C_2 = \{ZY^2 - X^3 = 0\}$ and $C_3 = \{X^3 + Y^3 + Z^3 = 0\}$. For example, you can do this with a computer algebra system.

Now, we want to write the Gauss map analytically. Take a point $p \in C_{reg} \cap \{Z \neq 0\}$ and suppose that in the affine coordinates x, y the curve has an analytic parametrization $t \mapsto (x(t), y(t))$, where t is a local coordinate at a point.

d) Show that in these coordinates the Gauss map can be written as $t \mapsto (p(t), q(t))$ with

$$p(t) = \frac{\dot{y}(t)}{y(t)\dot{x}(t) - x(t)\dot{y}(t)}, \qquad q(t) = -\frac{\dot{x}(t)}{y(t)\dot{x}(t) - x(t)\dot{y}(t)}$$

Now, let $C \subseteq \mathbb{P}^2$ be a plane curve of degree d and $C^\vee \subseteq \mathbb{P}^{2*}$ the dual curve. We can then take the dual of this second curve to get another curve $(C^\vee)^\vee \subseteq (\mathbb{P}^{2*})^*$. A fundamental fact about these is:

e) **Biduality Theorem:** Under the natural identification $(\mathbb{P}^{2*})^* \cong \mathbb{P}^2$ we have an identification $(C^*)^* \cong C$. Moreover, the composition of Gauss maps $C_{reg} \to C^\vee$ and $C^\vee_{reg} \to (C^\vee)^\vee \cong C$ is the identity. Prove both statements.

In particular, we can use this fact to compute the degree of the dual curve $C^\vee$, when C is a *smooth plane curve*.

f) When C is smooth of degree d, show that $\deg(C^\vee) = d(d-1)$.

g) When C is smooth of degree d, show that $C^\vee$ is never smooth, unless $d = 2$. For example, consider the curve $C_3 = \{X^3 + Y^3 + Z^3 = 0\}$ and the dual $C_3^\vee$. Can you interpret the singularities of $C_3^\vee$ geometrically?