Riemann Surfaces and Algebraic Curves
WiSe 2020/21

Exercise Sheet 9

These exercises will be discussed on February 3

Exercise 9.1 (General divisors) Let $d \geq 1$ be an integer

a) For any d points $P_1, \ldots, P_d \in \mathbb{P}^r$ we define the $\operatorname{span}(P_1, \ldots, P_d) \subseteq \mathbb{P}^r$ as the smallest linear subspace containing all the points. Now consider the d-fold cartesian product $\mathbb{P}^r \times \cdots \times \mathbb{P}^r$ and prove that the following subset V is open in it:

$$V = \{(P_1, \ldots, P_d) \mid \dim \operatorname{span}(P_1, \ldots, P_d) = d - 1\}.$$

Now, let X be a non-hyperelliptic curve, hence of genus g at least three, and consider the cartesian product X^d.

b) Show that for $d \leq g$ there is a non-empty open subset U of X^d such that $h^0(X, \sum_{i=1}^d P_i) = 1$ and $h^0(X, K - \sum_{i=1}^d P_i) = g - d$ for all $(P_1, \ldots, P_d) \in U$.

c) Show that for $d \geq g$ there is a non-empty open subset U of X^d such that $h^0(X, \sum_{i=1}^d P_i) = d + g - 1$ and $h^0(X, K - \sum_{i=1}^d P_i) = 0$ for all $(P_1, \ldots, P_d) \in U$.

Exercise 9.2 (Secant spaces) Let X be an algebraic curve of genus 2 and let D be a divisor on X of degree 6 so that $h^0(X, D) = 5$. Let $\iota \colon X \to \mathbb{P}^4$ be the corresponding map (determined up to linear changes of coordinates by the choice of basis of $H^0(X, D)$). Show that for every secant line $L = \operatorname{span}(P + Q) \subset \mathbb{P}^4$ to $\iota(X)$ there exists another secant line $L' = \operatorname{span}(P' + Q')$ such that $L \cap L' \neq \emptyset$.

Exercise 9.3 (Clifford's Theorem) Let X be a compact Riemann surface of genus g and let K be a canonical divisor on X.

a) For two divisors D_1, D_2 on X, define $\min\{D_1, D_2\} = \sum_{P \in X} \min\{(D_1)_P, (D_2)_P\} P$ and analogously $\max\{D_1, D_2\}$. Show that

$$h^0(X, D_1) + h^0(X, D_2) \leq h^0(X, \min\{D_1, D_2\}) + h^0(X, \max\{D_1, D_2\}).$$

b) Let D be a divisor on X such that both $h^0(X, D)$ and $h^0(X, K - D)$ are positive. Conclude that

$$h^0(X, D) + h^0(X, K - D) \leq 1 + g.$$

c) Let D be a divisor on X such that both $h^0(X, D)$ and $h^0(X, K - D)$ are positive. Show that

$$2(h^0(X, D) - 1) \leq \deg(D).$$