Exercise 8.1: a) If $X \cong \mathbb{P}^1$ then it has genus 0.

If X has genus 0, choose any divisor D of degree 1. Then $\deg D \geq 2g+1$ so we know from the lectures that D is very ample and moreover $h^0(D) = 2$. So D defines an embedding $X \to \mathbb{P}^1$ which must be an isomorphism.

b) Recall that p is a base point of K iff $h^0(K_X - p) = h^0(K_X) = g$. Hence iff

$$h^0(p) = 1 + 1 - g + h^0(K_X - p) = 2$$

However in this case there is a nonconstant merom. function $f \in H^0(X, p)$ and $f: X \to \mathbb{P}^1$ has degree 1. Equivalently, the linear system $V = H^0(X, p)$ defines a map $\varphi: X \to \mathbb{P}^1$ of degree 1.

c) Recall that K does not separate $p, q \in X$ (possibly coincident) iff $h^0(K_X - p - q) > h^0(K_X) - 2 = g - 2$

Hence iff $h^0(p+q) = 2 + 1 - g + h^0(K_X - p - q) \geq 2$.

But then there is a nonconstant rational function $f \in H^0(X, p+q)$ and the induced map $f: X \to \mathbb{P}^1$ has degree at most 2. If it has degree 1, then X is $\mathbb{P}^1$ which is also hyperelliptic.

Conversely, if X is hyperelliptic, then K is not very ample because of Exercise 8.3.(b).

Exercise 8.2 ; a) Choose two local coordinates z, w and let $z = \phi(w)$ the change of coords. Then

$$\omega_i = f_i(z)dz = h_i(w)dw \quad \text{and}$$

$$f_i(z)dz = f_i(\phi(w))\, d\phi(w) = f_i(\phi(w))\dot{\phi}(w)\, dw$$

so $h_i(w) = f_i(\phi(w)) \cdot \dot{\phi}(w)$. Hence

$$[h_1(w), .., h_g(w)] =$$

$$= [f_1(\phi(w))\cancel{\dot{\phi}(w)}, .., f_g(\phi(w))\cancel{\dot{\phi}(w)}]$$

$$= [f_1(z), .., f_g(z)].$$

Furthermore, if we choose different bases of $H^0(X, \omega_X)$ it is clear that this induces a linear change of coords on $\mathbb{P}^{g-1}$.

(b) Let $K = \operatorname{div}(\omega_0)$ be a canonical divisor. Recall that we have an isomorphism

$$H^0(X, \omega_X) \to H^0(X, K) \quad \omega \mapsto \left(\frac{\omega}{\omega_0}\right)$$

and if in local coordinates we have $\omega_i = f_i(z)dz$ then $\left(\frac{\omega_i}{\omega_0}\right) = \frac{f_i(z)}{f_0(z)}$, so the two maps coincide

$$[\omega_1, .., \omega_g] = [f_1(z), .., f_g(z)]$$

$$\left[\frac{\omega_1}{\omega_0}, .., \frac{\omega_g}{\omega_0}\right] = \left[\frac{f_1(z)}{f_0(z)}, .., \frac{f_g(z)}{f_0(z)}\right]$$

Moreover K is bpf because X has genus >0. So we know that

$$\{\varphi^* H \mid H \subseteq \mathbb{P}^{g-1} \text{ hyperplane}\} = |K|$$

and any effective divisor in $|K|$ is an effective canonical divisor: $E = K + \operatorname{div}(f)$ then

$$E = \operatorname{div}(f \cdot \omega_0).$$

(c) p is a base point for K iff

$$\operatorname{ord}_p f + \operatorname{ord}_p K > 0 \quad \forall f \in H^0(X, K)$$

which means $\operatorname{ord}_p(f \cdot \omega_0) > 0$. But since we have the isomorphism

$$H^0(X, K) \longrightarrow H^0(X, \omega_X) \quad f \longmapsto f \cdot \omega_0$$

the conclusion follows.

(d) Recall the pullback of differentials:
we have charts U_i on X_1 and V_i on X_2 with coordinates z_i and w_i respectively s.t.

$$f|_{U_i} = f_i : U_i \longrightarrow V_i \qquad f_i(z_i) = w_i$$

Then if $\omega = (h_i(w_i)\, dw_i)$ is a differential on X_2 we can pull it back to a differential

$$f^* \omega = (h_i(f_i(z_i))\, \dot{f}_i(z_i)\, dz_i)$$

on X_1. One can show that

$f^*: H^0(X_2, \omega_{X_2}) \to H^0(X_1, \omega_{X_1})$ are inverse to each other.
$(f^{-1})^*: H^0(X_1, \omega_{X_1}) \to H^0(X_2, \omega_{X_2})$

Now let $\omega_1, \dots, \omega_g$ be a basis of $H^0(X_2, \omega_{X_2})$ and take a corresponding basis $f^*\omega_1, \dots, f^*\omega_g$ of $H^0(X_1, \omega_{X_1})$. Then we have the two canonical maps $\varphi_1: X_1 \to \mathbb{P}^{g-1}$, $\varphi_2: X_2 \to \mathbb{P}^{g-2}$ and in the local coordinates of before we have

$$\varphi_1(f^{-1}(w_i)) = [h_1(w_i)\cancel{f_i'(z_i)}, \dots, h_g(w_i)\cancel{f_i'(z_i)}]$$
$$= [h_1(w_i), \dots, h_g(w_i)] = \varphi_2(w_i).$$

So we get what we want. If we choose different bases of $H^0(X_1, \omega_{X_1})$, $H^0(X_2, \omega_{X_2})$ we can go back to this case by a linear change of coords on $\mathbb{P}^{g-1}$.

Exercise 8.3 : a) X has two charts

$$X_0 = \{ y^2 = f(x) \} \qquad x = 1/z$$
$$X_1 = \{ w^2 = \tilde{f}(z) \} \qquad y = w/z^{g+1}$$

$$f(x) = \prod (x - \lambda_i) \qquad \tilde{f}(z) = \prod (1 - \lambda_i z)$$

The differentials $1/y \, dx, \dots, x^{g-1}/y \, dx$ are a priori only meromorphic and only defined on X_0 we should extend them to the whole X so that they are holomorphic. To extend them to X, we do

$$\frac{x^i}{y} dx = \frac{z^{-i}}{w z^{-g-1}} dz^{-1} = -\frac{z^{g+1-i}}{w} \cdot \frac{1}{z^2} dz = -\frac{z^{g-1-i}}{w} dz$$

so we get meromorphic differentials on X. We check that they are holomorphic. First on X_0: we have $\frac{x^i}{y} dx$ is holomorphic when $y \neq 0$ (notice also that at those pts x is a local coord)

On the other pts we use the relation

$$y^2 - f(x) = 0 \Rightarrow 2y\,dy - f_x dx = 0$$
$$\Rightarrow 1/y \, dx = 2/f_x \, dy$$

to find the equivalent form $\frac{x^i}{y} dx = \frac{2x^i}{f_x} dy$, which is holomorphic, because there are no pts where $y = 0, f_x = 0$, since X is smooth.

On the other chart, we can do something similar.

Hence $\frac{1}{y}dx, \dots, \frac{x^{g-1}}{y}dx$ are in $H^0(X, \omega_X)$.

To show that they are a basis, it is enough to prove linear independence, since $h^0(X, \omega_X) = g$.

Suppose that $\sum_{i=0}^{g-1} \lambda_i \frac{x^i}{y} dx = 0$, then $\sum_{i=0}^{g-1} \lambda_i x^i = 0$

on X. Meaning that $\{y^2 = f(x)\}$ is contained inside $\{\sum \lambda_i x^i = 0\}$ which is absurd (unless $\lambda_i = 0$).

Finally we see that the canonical map factors as

$$\begin{tikzcd}
X \arrow[rr, "{[1/y, x/y, \dots, x^{g-1}/y]}"] \arrow[dr, "x"', "2:1"] & & \mathbb{P}^{g-1} \\
& \mathbb{P}^1 \arrow[ur, hook, "{[1, x, \dots, x^{g-1}]}"'] &
\end{tikzcd}$$

b) Suppose $X = \{F(X, Y, Z) = 0\}$ where F is of degree d. Then $f(x, y) = F(x, y, 1)$ so $x = \frac{X}{Z}, y = \frac{Y}{Z}$. We look at the differential $\frac{x^a y^b}{f_y} dx$

This is a priori defined only on the chart $\{z \neq 0\}$ but we see it is holomorphic here: indeed

$f(x, y) = 0 \Rightarrow f_x dx + f_y dy = 0 \Rightarrow$

$\Rightarrow \frac{x^a y^b}{f_y} dx = - \frac{x^a y^b}{f_x} dy$ and one of f_x, f_y is nonzero on X.

How do we extend these to differentials on the whole X? For example, take the other chart $\{Y \neq 0\}$. Then here we have coordinates $x' = \frac{X}{Y}$, $z' = \frac{Z}{Y}$ and the change of coords is given by:

$$x = \frac{X}{Z} = \left(\frac{X}{Y}\right)\left(\frac{Z}{Y}\right)^{-1} = \frac{x'}{z'}$$

$$y = \frac{Y}{Z} = 1/z'$$

so that

$$f(x,y) = F(x,y,1) = F\left(\frac{x'}{z'}, 1/z', 1\right) = \left(\frac{1}{z'}\right)^d F(x', 1, z')$$

$$f_y(x,y) = F_Y(x,y,1) \qquad = \left(1/z'\right)^{d-1} F_Y(x',1,z')$$

so we can write

$$-\frac{x^a y^b}{f_y(x,y)} dy = -\frac{(x')^a}{(z')^{a+b-d+1} F_Y(x',1,z')} d\left(\frac{1}{z'}\right)$$

$$= \frac{(x')^a}{(z')^{a+b-(d-3)} F_Y(x',1,z')} dz'$$

and then we check that these are holomorphic.

This shows that the differentials

$$\frac{x^a y^b}{f_y} dx \in H^0(X, \omega_X)$$

To show they are a basis, it is enough to prove that they are linearly independent, because

$$h^0(X,\omega_X) = g = \frac{(d-1)(d-2)}{2} = \#\{x^a y^b \mid a+b\leq d-3\}$$

If they are linearly dependent then this means that $\sum \lambda_{ab} x^a y^b = 0$ so there is a polynomial $g(x,y) = \sum \lambda_{ab} x^a y^b$ of degree at most $d-3$ such that

$$\{f(x,y)=0\} \subseteq \{g(x,y)=0\}$$

which is impossible, because $\deg f = d$.

To conclude, the $d-3$ Veronese embedding of $\mathbb{P}^2$ is

$$\begin{array}{ccc} \mathbb{P}^2 & \hookrightarrow & \mathbb{P}^{N_{d-3}} \\ [x,y,1] & \longmapsto & [x^a y^b]_{a+b\leq d-3} \end{array}$$

and if we restrict this to X we get exactly the canonical map induced by our basis.

(c) Suppose X is a smooth plane curve of degree $d\geq 4$. Then (b) shows that the canonical map is an embedding. On the other hand (a) shows that the canonical map for hyperelliptic curves is not an embedding.