Riemann Surfaces and Algebraic Curves

WiSe 2020/21

Exercise Sheet 8

These exercises will be discussed on January 27

Exercise 8.1 (Consequences of Riemann-Roch)

a) Show that a compact Riemann surface X has genus zero if and only if $X \cong \mathbb{P}^1$.

b) Show that a canonical divisor on a Riemann surface X is always base-point-free, unless $X \cong \mathbb{P}^1$.

c) Show that a canonical divisor on a Riemann surface X is very ample if and only if X is not hyperelliptic, meaning that there is no degree two map $X \to \mathbb{P}^1$.

Exercise 8.2 (The canonical map) Let X be a compact Riemann surface of positive genus and let $\omega_1, \dots, \omega_g$ be a basis of $H^0(X, \omega_X)$. We can define a *canonical map:*

$$\varphi\colon X \longrightarrow \mathbb{P}^{g-1}, \qquad p \mapsto [\omega_1(p), \dots, \omega_g(p)]$$

by taking for any point $p \in X$ a neighborhood where $\omega_i = f_i(z)dz$ and then setting $\varphi(p) = [f_1(z), \dots, f_g(z)]$.

a) Show that this is well-defined and unique, up to a linear change of coordinates.

b) Compare this map with the one induced by the complete linear system of a canonical divisor. In particular show that

$$\{\varphi^* H \mid H \subseteq \mathbb{P}^{g-1} \text{ hyperplane }\} = \{K \mid K \text{ canonical divisor }\}$$

c) Prove that a canonical divisor is base-point-free if and only if for every point $p \in X$ there is a differential form $\omega \in H^0(X, \omega_X)$ such that $\omega(p) \neq 0$.

d) Let $f\colon X_1 \to X_2$ an isomorphism of compact Riemann surfaces and let $\varphi_i\colon X_i \to \mathbb{P}^{g-1}$ be a canonical map for X_i. Show that there is a linear change of coordinates $F\colon \mathbb{P}^{g-1} \to \mathbb{P}^{g-1}$ such that $F \circ \varphi_1 = \varphi_2 \circ f$.

Exercise 8.3 (Differentials)

a) Let X be an hyperelliptic curve with affine model $\{y^2 = f(x)\}$, where $f(x)$ is a polynomial of degree $2g + 2 \geq 4$. Show that a basis of $H^0(X, \omega_X)$ is given by

$$\frac{1}{y}dx, \frac{x}{y}dx, \dots, \frac{x^{g-1}}{y}dx.$$

In particular you should understand how to extend this to the whole curve. Conclude that canonical map $X \to \mathbb{P}^{g-1}$ is a double cover of X onto a rational normal curve of degree $g - 1$.

b) Let $X \subseteq \mathbb{P}^2$ be a smooth plane curve of degree d, with an affine model $\{f(x, y) = 0\}$. Show that a basis of $H^0(X, \omega_X)$ is given by

$$\frac{x^a y^b}{f_y}dx, \qquad a \geq 0, b \geq 0, a + b \leq d - 3$$

In particular observe that the canonical map can be realized as the composition

$$X \hookrightarrow \mathbb{P}^2 \hookrightarrow \mathbb{P}^{N_{d-3}}$$

where the second map is the Veronese embedding of degree $d - 3$.

c) Show that no smooth plane curve of degree $d \geq 4$ is hyperelliptic.