Exercise 7.1 : (a) $V \subseteq H^0(X, D)$ a base-point-free linear system, $f_0, \dots, f_r$ a basis of V and

$$\varphi : X \longrightarrow \mathbb{P}^r \text{ the induced map}$$

Prove that $\{\varphi^* H \mid H \text{ hyperplane}\} = \{\operatorname{div}(f) + D \mid f \in V\}$.

sol : $H = \{a_0 X_0 + \dots + a_r X_r\}$ hyperplane.

We will show that

$$\boxed{\varphi^* H = \operatorname{div}\left(\sum a_i f_i\right) + D.}$$

Let $p \in X$. Since V is bpf, there is one f_i such that $\operatorname{ord}_p(f_i) + D_p = 0$. We assume $f_i = f_0$.
Then observe that the f_i/f_0 are all holomorphic around p:

$$\operatorname{ord}_p(f_i/f_0) = \operatorname{ord}_p(f_i) - \operatorname{ord}_p(f_0) = \operatorname{ord}_p(f_i) - D_p \geq 0$$

So around p we can write $\varphi = [1, \frac{f_1}{f_0}, \dots, \frac{f_r}{f_0}]$ with the f_i/f_0 holomorphic. Then we see that $\varphi(p) \in \{X_0 \neq 0\}$ and then

$$(\varphi^* H)_p = \operatorname{ord}_p\left(\sum a_i \frac{f_i}{f_0}\right)$$
$$= \operatorname{ord}_p\left(\sum a_i f_i\right) - \operatorname{ord}_p(f_0)$$
$$= \operatorname{ord}_p(f) + D_p.$$

$\square$

Exercise 7.2 : (a) Show that $\varphi : X \longrightarrow \mathbb{P}^r$ is injective iff Δ separates points. Express this in terms of functions in V.

Sol : Let $p, q \in X$ be distinct. We see that $\varphi(p) \neq \varphi(q)$ iff there is an hyperplane $H \subseteq \mathbb{P}^r$ passing through $\varphi(p)$ and not $\varphi(q)$, i.e. $p \in \varphi^* H$, $q \notin \varphi^* H$. Since the divisors in Δ are exactly the $\varphi^* H$, we are done.

In terms of functions, this means there is a function $f \in V$ such that

$$\mathrm{ord}_p(f) + D_p > 0 \ , \ \mathrm{ord}_q(f) + D_q = 0 .$$

(b) Show that $d\varphi_p : T_p X \longrightarrow T_p \mathbb{P}^r$ is injective iff there exists $E \in \Delta$ such that $E_p = 1$.

Express this in terms of functions in V.

Sol : as in a previous exercise, we can assume that $\mathrm{ord}_p(f_0) + D_p = 0$. Hence around p we can write the map as $\varphi = [1, f_1/f_0, \dots, f_r/f_0]$ where the f_i/f_0 are holomorphic. Moreover up to a linear change of coordinates (which does not change the linear system), we can assume $f_i/f_0 (p) = 0$, i.e. $\mathrm{ord}_p(f_i) + \underbrace{D_p}_{-\mathrm{ord}_p(f_0)} > 0$ $i = 1, \dots, r$.

If z is a local coordinate around p, then $d\varphi_p$ is injective <u>iff</u> there exists an i s.t.

$$\frac{\partial}{\partial z}\left(\frac{f_i}{f_0}\right)(p) \neq 0, \text{ i.e. } \mathrm{ord}_p(f_i) + \underbrace{D_p}_{= -\mathrm{ord}_p(f_0)} = 1.$$

But then $E = \mathrm{div}(f_1) + D \in \Lambda$ and $E_p = 1$. The converse is similar.

In terms of functions in V, this means there exists a function $f \in V$ s.t.

$$\mathrm{ord}_p(f) + D_p = 1.$$

$\square$

(C) Suppose that φ is induced by a complete linear system $H^0(X,D)$.

φ embedding $\iff$ $h^0(X, D-p-q) = h^0(X,D)-2$ $\forall\, p,q \in X$ (possibly coincident).

Sol: ($\Rightarrow$) • $p,q \in X$ distinct.

There is a divisor containing q and not p, so not every divisor contains p.

$$|D-p| \subsetneq |D|$$

Then there is a divisor containing p and not q. So

$$|D-p-q| \subsetneq |D-p| \subsetneq |D|$$

Hence

$$h^0(D-p-q) = h^0(D-p)-1$$
$$= h^0(D)-1-1 = h^0(D)-2.$$

• $p \in X$. As before, $|D-p| \subsetneq |D|$ and there is a divisor of order 1 at p, hence

$$|D-2p| \subsetneq |D-p| \subsetneq |D|$$

So $h^0(D-2p) = h^0(D-p) - 1 = h^0(D) - 2$.

($\Leftarrow$) This is similar.

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(d) $X = \mathbb{P}^1$; D is very ample iff $\deg D \geq 1$.
$X =$ torus : D is very ample iff $\deg D \geq 3$.

<u>Sol</u> : • $X = \mathbb{P}^1$, $h^0(X, D) = \deg D + 1$ for $\deg D \geq 0$

. $h^0(X, D-p-q) = \deg(D-p-q) + 1$ if $\deg D \geq 2$

$= \deg D + 1 - 2$

$= h^0(X, D) - 2$

· if $\deg D = 1$ then

$h^0(X, D-p-q) = 0 = 2 - 2$

$= h^0(X, D) - 2$.

Say $\deg D = d$. $D \sim d \cdot \infty$

$H^0(\mathbb{P}^1, d \cdot \infty) = \langle 1, x, \dots, x^d \rangle$ the map is

$$\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^d$$
$$x \longmapsto [1, x, x^2, \dots, x^d]$$

rational normal curve of degree d.

- $X =$ complex torus: $h^0(X, D) = \deg D$ for $\deg D \geq 1$

so if $\deg D \geq 3$ then

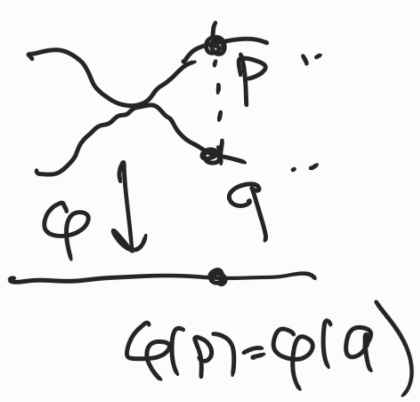
$$h^0(X, D-p-q) = \deg(D-p-q)$$
$$= \deg(D) - 2$$
$$= h^0(X, D) - 2$$

If $\deg D = 1$: $h^0(X, D) = 1$

$h^0(X, D-p-q) = 0$ because neg. degree

If $\deg D = 2$: $H^0(X, D) = \langle f_0, f_1 \rangle$

$\varphi : X \longrightarrow \mathbb{P}^1$ the map cannot be an embedding because otherwise $X \cong \mathbb{P}^1$ but they have different genera. For example, the map has to have degree ≥ 2.



∴ p, q are not separated by φ

$h^0(X, D-p-q) \neq h^0(X, D) - 2$.

Exercise 7.3 : (a) In any way you do it, you can see

$$\left(\frac{\partial^3 \log\vartheta}{\partial z^3}\right)^2 + 4\left(\frac{\partial^2 \log\vartheta}{\partial z^2}\right)^3 = \frac{\cdots\cdots}{\vartheta^4}$$

so it is a function with a pole of order at most 4.

A basis of $H^0(X, 4p)$ is given by

$$\left\langle 1, \underset{\substack{\downarrow \text{pole} \\ 2}}{\frac{\partial^2 \log\vartheta}{\partial z^2}}, \underset{\substack{\downarrow \text{pole} \\ 3}}{\frac{\partial^3 \log\vartheta}{\partial z^3}}, \underset{\substack{\downarrow \text{pole} \\ 4}}{\left(\frac{\partial^2 \log\vartheta}{\partial z^2}\right)^2} \right\rangle$$

So we must be able to write

$$\left(\frac{\partial^3 \log\vartheta}{\partial z^3}\right)^2 + 4\left(\frac{\partial^2 \log\vartheta}{\partial z^2}\right)^3 =$$

$$= a\cdot\left(\frac{\partial^2 \log\vartheta}{\partial z^2}\right)^2 + b\left(\frac{\partial^2 \log\vartheta}{\partial z^2}\right) + c + d\cdot\frac{\partial^3 \log\vartheta}{\partial z^3}$$

(b) Since ϑ is even, $\frac{\partial \log\vartheta}{\partial z} = \frac{\vartheta'}{\vartheta}$ is odd

so $\frac{\partial^2 \log\vartheta}{\partial z^2}$ is even and $\frac{\partial^3 \log\vartheta}{\partial z^3}$ is odd,

and since in the equation the LHS is even the RHS must be even as well, so $d = 0$.

(c) Then we see that

$$\varphi(X) \subseteq \{ y^2 = -4x^3 + ax^2 + bx + c \} = E$$

Since φ is an embedding, it gives an isomorphism $X \cong \varphi(X)$. This image needs to be a smooth plane curve C.

The curve C has degree 3, indeed if $H \subseteq \mathbb{P}^2$ is a line, we know that $\varphi^* H$ has degree 3 (because it is in the linear system $|3p|$), hence H intersects C in 3 pts (with multiplicity). But then $C \subseteq E$ and both have degree 3, thus $C = E$.

Aside: Weierstraß $\wp$-function

If $X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, then the associated Weierstrass $\wp$-function is:

$$\wp(z) = \frac{1}{z^2} + \sum_{(n,m) \neq 0} \left(\frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right)$$

this is invariant under the lattice $\mathbb{Z}+\tau\mathbb{Z}$ so it defines a meromorphic function on X with a pole of order 2 at 0.

Hence, $\wp(z-1/2-1/2\tau)$ has a pole of order 2 at $p=1/2+1/2\tau$. Thus $\wp(z-1/2-1/2\tau)\in H^0(X,2p)$ and this means that

$$\wp(z-1/2-1/2\tau)=a\cdot\frac{\partial^2\log\vartheta}{\partial z^2}(z)+b$$

By looking at the coefficient of $\frac{1}{(z-1/2-1/2\tau)^2}$ we can see that $a=-1$. So

$$\wp(z-1/2-1/2\tau)=-\frac{\partial^2\log\vartheta}{\partial z^2}(z)+b.$$