Exercise Sheet 7

These exercises will be discussed on January 20

Exercise 7.1 (Linear systems and maps to projective space) Let X be a compact Riemann surface and D a divisor on X.

a) Let $V \subseteq H^0(X,D)$ be a base-point-free subspace of dimension $r+1$ and $\Lambda = \{\operatorname{div}(f) + D \mid f \in V\}$ the corresponding linear system. Let $f_0, \dots, f_r$ be a basis of V and consider the map $\varphi\colon X \longrightarrow \mathbb{P}^r$ given by $\varphi = [f_0, f_1, \dots, f_r]$. Show that $\{\varphi^* H \mid H \subseteq \mathbb{P}^r \text{ hyperplane }\} = \Lambda$.

b) What happens if V has base points? More generally, we define the base locus of V as the largest effective divisor B such that $B \leq D$ for all $D \in \Lambda$. Show that the map $\varphi = [f_0, \dots, f_r]$ is still defined, but now $\{\varphi^* H \mid H \subseteq \mathbb{P}^r \text{ hyperplane }\} = \Lambda - B$, where $\Lambda - B = \{D - B \mid D \in \Lambda\}$. So the correspondence breaks down when the linear system has base points.

Exercise 7.2 (Projective embeddings) Let X be a compact Riemann surface, $V \subseteq H^0(X,D)$ be a base-point-free subspace of dimension $r+1$ and Λ the corresponding linear system. Choose a basis of V and define the map $\varphi = [f_0, f_1, \dots, f_r]$. When is this map an embedding?

a) Show that the map φ is injective if and only if Λ separates points. This means that for any two distinct points $p, q \in X$ there is $D \in \Lambda$ such that $p \in D$ and $q \notin D$. Express this condition also in terms of functions in V.

b) Show that the differential of φ is everywhere injective if and only if Λ separates tangent vectors. This means that for each point p there is $D \in \Lambda$ such that $\operatorname{ord}_p(D) = 1$. Express this condition also in terms of functions in V.

c) Suppose that φ is defined by the complete linear system $H^0(X,D)$. Prove that it is an embedding if and only if

$$h^0(X, D - p - q) = h^0(X, D) - 2$$

for any two points $p, q \in X$ possibly coincident. In this case we say that D is *very ample.*

d) Show that a divisor D on $\mathbb{P}^1$ or on a complex torus X is very ample if and only if $\deg D \geq 1$ or $\deg D \geq 3$ respectively. Write down the map induced by a divisor of degree d on $\mathbb{P}^1$ (you can choose the divisor).

Exercise 7.3 (Complex tori and elliptic curves) Take $\tau \in \mathcal{H}$ and the corresponding complex torus $X = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. We also take the theta function $\theta(z) = \theta_\tau(z)$ and its zero $p = \left[\frac{1}{2} + \frac{1}{2}\tau\right]$. The embedding induced by $D = 3p$ is

$$\varphi\colon X \hookrightarrow \mathbb{P}^2, \qquad \varphi = \left[1, \frac{\partial^2 \log\theta}{\partial z^2}, \frac{\partial^3 \log\theta}{\partial z^3}\right]$$

a) Show that $\left(\frac{\partial^3 \log\theta}{\partial z^3}\right)^2 + 4\left(\frac{\partial^2 \log\theta}{\partial z^2}\right)^3 \in H^0(X, 4p)$, possibly via computer algebra. Deduce that

$$\left(\frac{\partial^3 \log\theta}{\partial z^3}\right)^2 + 4\left(\frac{\partial^2 \log\theta}{\partial z^2}\right)^3 = a\left(\frac{\partial^2 \log\theta}{\partial z^2}\right)^2 + b\cdot\frac{\partial^2 \log\theta}{\partial z^2} + c + d\cdot\frac{\partial^3 \log\theta}{\partial z^3}$$

for certain $a, b, c, d \in \mathbb{C}$.

b) Looking at the involution $\sigma\colon \mathbb{C} \to \mathbb{C}, \sigma(z) = -z$, deduce that $d = 0$.

c) Conclude that $\varphi\colon X \hookrightarrow \mathbb{P}^2$ gives an isomorphism of X with the elliptic curve $E = \{y^2 = -4x^3 + ax^2 + bx + c\}$.