Exercise 6.1 : X = compact Riemann surface

$f : X \longrightarrow \mathbb{P}^1$ of degree d, $D = \mathrm{div}_\infty(f)$.

(a) $g \in \mathbb{C}(X)$. Then there exists a polynomial $P(f)$ in f s.t. $P(f)\cdot g \in H^0(X, m_0 D)$.

Sol: Write $D = n_1 q_1 + \dots + n_r q_r$. It's enough to find a polynomial $P(f)$ s.t. $P(f)\cdot g$ has poles only at $q_1, \dots, q_r$. Indeed, if

$$\mathrm{div}_\infty(P(f)g) = n_1' q_1 + \dots + n_r' q_r$$

then choose m_0 s.t. $m_0 \cdot n_i \geq n_i'$ for all i. Then $P(f)\cdot g \in H^0(X, m_0 D)$.

Let $\mathrm{div}_\infty(g) = n_1 p_1 + n_2 p_2 + \dots + n_r p_r$

Suppose first that none of these points appear in D. Then $f(p_1), \dots, f(p_r) \in \mathbb{A}^1$. Let's consider the polynomial $P(x) = (x - f(p_1))^{n_1} \cdots (x - f(p_r))^{n_r}$ and the rational function $P(f)\cdot g$. We see that $P(f)g$ has poles only on the support of D, so we are happy. The case where some of the points p_i appear in D is even easier (why?).

(b) Let $g_1, \dots, g_k \in \mathbb{C}(X)$ that are $\mathbb{C}(f)$-lin indep. Then we can assume $g_1, \dots, g_k \in H^0(X, m_0 D)$ for a certain m_0.

sol: if $g_1, \dots, g_k$ are $\mathbb{C}(f)$-lin. indep the same is true if we multiply them by a polynomial in f. By point (a) we have polys $P_i(f) g_i \in H^0(X, m_i D)$, then choose m_0 as the largest m_i. $\square$

(c) $f^i g_j$, $i \leq m$ are $\mathbb{C}$-lin. indep in $H^0(X, (m_0+m)D)$. Hence $h^0(X, (m_0+m)D) \geq (m+1)k$.

sol: it is clear that $f^i g_j \in H^0(X, (m_0+m)D)$.
To check linear independence:

$$0 = \sum_{i,j} a_{ij} f^i g_j = \sum_j \left(\sum a_{ij} f^i \right) g_j$$

$$= \sum_j P_j(f) g_j$$ Hence $P_j(f) = 0$ since the g_i are $\mathbb{C}(f)$-lin independent.

Then $P(f) = 0$ implies that the $a_{ij} = 0$ (we need that $1, f, f^2, \dots, f^m$ are lin. indep. why?). $\square$

(d) Conclude that $[\mathbb{C}(X) : \mathbb{C}(f)] \leq d$.

sol: Our basic estimates gives us

$$(m+m_0)d \geq h^0(X, (m_0+m)D) \geq (m+1)k \quad \square$$

Exercise 6.2: $X = \{y^2 = f(x)\}$ $\deg f(x) = 2g+1$

(a) We know that $[\mathbb{C}(X) : \mathbb{C}(x)] \leq 2$.
If we can show that $[\mathbb{C}(x,y) : \mathbb{C}(x)] = 2$
we are done. We will show that $1, y$ are
$\mathbb{C}(x)$-linearly independent. Equivalentely
$y \notin \mathbb{C}(x)$. Suppose $y \in \mathbb{C}(x)$. Then

$$y = \frac{a(x)}{b(x)} \qquad a, b \text{ coprime}$$

Hence
$$f(x) = y^2 = \frac{a(x)^2}{b(x)^2}$$

absurd, because $f(x)$ has simple zeroes and
$\left(\frac{a(x)}{b(x)}\right)^2$ has double zeroes. $\square$

(b) From (a) we know that $\mathbb{C}(X) = \mathbb{C}(x,y)$
$= \mathbb{C}(x)[y]/(y^2 - f(x))$. Consider the natural
involution $\sigma : X \to X$ $(x,y) \mapsto (x,-y)$.
which acts naturally on the space of meromorphic
functions $\sigma : \mathbb{C}(X) \to \mathbb{C}(X)$, $g(x,y) \mapsto \sigma^* g = g(x,-y)$
Since $\sigma(p_\infty) = p_\infty$, if g has poles only at p_∞
the same is true of $\sigma^*(g)$. Now suppose that

$$g = a(x)+b(x)y \quad a,b \in \mathbb{C}(x)$$

has poles only at P_∞. Then

$$\sigma^*(g) = a(x)-b(x)y$$

has poles only at P_∞. Then

$$g+\sigma^*(g) = 2\cdot a(x)$$
$$g-\sigma^*(g) = 2\, b(x)\, y$$

have poles only at P_∞. This means that both $a(x), b(x)$ are polynomials. $\square$

(c) Suppose $\deg f = 5$, so $g = 2$.

First we can look more in general at the possible orders of a pole of a polynomial

$$A(x)+B(x)y \quad \text{at } P_\infty$$

First we observe that $\mathrm{ord}_{P_\infty}(x) = -2 \quad \mathrm{ord}_{P_\infty}(y) = -5$

Now suppose $\deg A = a$, $\deg B = b$. Then

$$\mathrm{ord}_{P_\infty} A(x) = -2a$$
$$\mathrm{ord}_{P_\infty} B(x)y = -5-2b$$

and since these two are always distinct (one odd one even) we see that

$$\mathrm{ord}_{P_\infty}(A(x)+B(x)y) = \min\{-2a, -5-2b\}$$

So the possible orders are: $0, -2, -4, -5, -6, -7, -8, \dots$

Now we compute:

- $h^0(X, p_\infty) \leq 1$: basis 1
- $h^0(X, 2p_\infty) \leq 2$: basis $1, x$
- $h^0(X, 3p_\infty) = h^0(X, 2p_\infty)$: because there are no functions of order -3. Hence

  basis $1, x$
- $h^0(X, 4p_\infty) \leq h^0(X, 3p_\infty) + 1$:

  basis $1, x, x^2$
- $h^0(X, 5p_\infty) \leq h^0(X, 4p_\infty) + 1$:

  basis $1, x, x^2, y$

Exercise 6.3: Let $\tau \in \mathcal{H}$. Recall the theta function

$$\vartheta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z}$$

It vanishes on

$$p = \left[\tfrac{1}{2} + \tfrac{1}{2}\tau\right]$$

(a) Recall the quasiperiodicity property:

$$\vartheta(z + n + m\tau) = e^{-\pi i m^2 \tau - 2\pi i m z} \vartheta(z)$$

Now let's take the logarithmic derivative:

$\frac{\partial^n \log \vartheta}{\partial z^n}$ this is a meromorphic function on $\mathbb{C}$

To check that this is meromorphic on $\mathbb{C}/\Lambda$ we check what happens:

$$\log \vartheta(z+n+m\tau) = \left(-\pi i m^2\tau - 2\pi i m z\right) + \log \vartheta(z)$$

$$\frac{\partial \log \vartheta}{\partial z}(z+n+m\tau) = -2\pi i m + \frac{\partial \log \vartheta}{\partial z}(z)$$

$$\frac{\partial^2 \log \vartheta}{\partial z^2}(z+n+m\tau) = \frac{\partial^2 \log \vartheta}{\partial z^2}(z)$$

and then the same holds for $n \geq 2$. To check the poles. We write the 2nd logaritmic derivative explicitly.

$$\frac{\partial \log \vartheta}{\partial z} = \frac{\vartheta'}{\vartheta} \qquad \frac{\partial^2 \log \vartheta}{\partial z^2} = \frac{\vartheta''\vartheta - (\vartheta')^2}{\vartheta^2}$$

So we see that $\frac{\partial^2 \log \vartheta}{\partial z^2}$ has a pole only at p and it has order 2, since ϑ has a simple zero at p.

$$\frac{\partial^2 \log \vartheta}{\partial z^2} = a \cdot z^{-2} + \dots$$

$$\frac{\partial^3 \log \vartheta}{\partial z^3} = -2a \cdot z^{-3} + \dots$$

so $\frac{\partial^n \log \vartheta}{\partial z^n} \in H^0(X, np)$

(b) Since the functions $1, \frac{\partial^2 \log \vartheta}{\partial z^2}, \ldots, \frac{\partial^n \log \vartheta}{\partial z^n}$ have different order of zeroes at p, they are linearly independent in $H^0(X, np)$ and $h^0(X, np) \leq n$, so they are a basis.

(c) Let $D = \sum n_i p_i$ be a divisor of positive degree $d > 0$ and let $q' = \sum n_i p_i$ be the sum in the torus. We can find another point q s.t. $d \cdot q = q'$. Then we know that $D \sim d \cdot q$. Then we can do the same thing of points (a), (b) with an appropriate translate of q.