MPI MiS Leipzig
Universität Leipzig
Dr. D. Agostini, Prof. Dr. R. Sinn

Riemann Surfaces and Algebraic Curves
WiSe 2020/21

Exercise Sheet 6

These exercises will be discussed on January 6

Exercise 6.1 (Meromorphic functions) Let X be a compact Riemann surface and $f \in \mathbb{C}(X)$ a nonconstant meromorphic function, inducing a map $f : X \to \mathbb{P}^1$ of degree d so that the divisor of poles $D = \mathrm{div}_\infty(f)$ is effective of degree d.

a) Let $g \in \mathbb{C}(X)$ be a rational function. Show that there is a polynomial $P(f)$ in f such that $P(f)g \in H^0(X, m_0 D)$ for a certain m_0. [*Hint:* look at the images of the poles of g along the map f].

b) Consider the field extension $\mathbb{C}(f) \subseteq \mathbb{C}(X)$. Suppose we have $g_1, \dots, g_k \in \mathbb{C}(X)$ which are $\mathbb{C}(f)$- linearly independent. Show that we can assume $g_1, \dots, g_k \in H^0(X, m_0 D)$ for a certain m_0.

c) Fix $m \geq 0$ and observe that the functions $f^i g_j$, $i \leq m$ are $\mathbb{C}$-linearly independent in $H^0(X, (m + m_0)D)$. Thus $h^0(X, (m + m_0)D) \geq (m + 1)k$.

d) Deduce that $k \leq d$, so that $[\mathbb{C}(X) : \mathbb{C}(f)] \leq d$.

Later we will see that $[\mathbb{C}(X) : \mathbb{C}(f)] = d$.

Exercise 6.2 (Meromorphic functions on hyperelliptic curves) Let X be a hyperelliptic Riemann surface with affine model given by $X = \{y^2 = f(x)\}$ where $f(x) \in \mathbb{C}[x]$ is a polynomial of odd degree with distinct roots. Let $x\colon X \to \mathbb{P}^1, (x, y) \mapsto x$ be the usual double cover. Observe that $x^{-1}(\infty)$, as a set, consists of an unique point p_∞.

a) Using the previous exercise, show that $\mathbb{C}(X) = \mathbb{C}(x, y)$. So every meromorphic function is a rational function in x and y.

b) Prove that the meromorphic functions with no poles outside of $\{p_\infty\}$ are exactly the polynomials in x and y. So they are described by the ring $\mathbb{C}[x, y]/(y^2 - f(x))$.

c) Now assume that $f(x)$ has degree 5 so that X has genus 2. Compute bases of the spaces $H^0(X, n \cdot p_\infty)$ for $n = 1, 2, 3, 4, 5$.

Exercise 6.3 (Linear systems on complex tori) Let $\tau \in \mathcal{H}$ be a complex number with positive imaginary part and let $X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ be the corresponding complex torus. We consider the theta function $\theta(z) = \theta_\tau(z)$ and its zero $p = \left[\frac{1}{2} + \frac{1}{2}\tau\right]$.

a) Show that the the logarithmic derivatives $\frac{\partial^n \log \theta}{\partial z^n}$ are meromorphic functions on X when $n \geq 2$.

b) For any fixed $n \geq 2$ show that the functions $1, \frac{\partial^2 \log \theta}{\partial z^2}, \frac{\partial^3 \log \theta}{\partial z^3}, \dots, \frac{\partial^n \log \theta}{\partial z^n}$ give a basis of $H^0(X, n \cdot p)$.

c) Conclude that if D is a divisor of positive degree on X, then

$$h^0(X, D) = \deg D.$$

In the next exercises we anticipate something from the lectures. First, a bit of notation. Let X be a compact Riemann surface and $\varphi\colon X \to \mathbb{P}^r$ a holomorphic map. Let $D \subseteq \mathbb{P}^r$ be a hypersurface such that $\varphi(X) \subsetneq D$. We want to define an effective divisor $\varphi^* D$ that counts the intersection points of $\varphi(X)$ and D with multiplicity. We do it as follows: for any $p \in X$ choose an affine chart $U \cong \mathbb{A}^r$ around $\varphi(p)$. If we choose a local coordinate z around p the map has the form

$$\varphi(z) = (\varphi_1(z), \varphi_2(z), \dots, \varphi_r(z)) \qquad \varphi_i(z) \text{ holomorphic}$$

In the affine chart, the hypersurface is described by one equation $D = \{f(x_1, \dots, x_r) = 0\}$. We can pullback this equation to the function $f(\varphi_1(z), \dots, \varphi_r(z))$ and then we define

$$(\varphi^* D)_p := \operatorname{ord}_p f(\varphi_1(z), \dots, \varphi_r(z)), \qquad \varphi^* D := \sum_{p \in X} (\varphi^* D)_p \cdot p$$

Exercise 6.4 (Basic facts on pullbacks) Let X be a compact Riemann surface.

a) Let $\varphi\colon X \to \mathbb{P}^r$ be a holomorphic map, and $D \subseteq \mathbb{P}^r$ a hypersurface. Show that $\varphi^* D$ is well defined, in the sense that it is independent of the choices we make.

b) Let $f\colon X \to \mathbb{P}^1$ be a surjective holomorphic map and $q \in \mathbb{P}^1$ a point, which we can consider as a hyperplane in $\mathbb{P}^1$. What is $f^* q$?

c) Suppose that $X \subseteq \mathbb{P}^2$ is a smooth plane curve and let $j\colon X \hookrightarrow \mathbb{P}^2$ be the embedding. Let $D = \{G = 0\}$ be another plane curve in $\mathbb{P}^2$. Show that the pullback divisor coincides with the intersection divisor: $j^* D = \operatorname{div} G$.

Exercise 6.5 (From map to projective spaces to linear systems) Let X be a compact Riemann surface.

a) Let $\varphi\colon X \to \mathbb{P}^r$ be a holomorphic map. Show that this can be written in the form

$$\varphi(p) = [1, f_1(p), \dots, f_r(p)]$$

where the f_i are meromorphic functions on X. In particular, you should understand how a map of this form is defined at the poles of the f_i. Moreover, observe that the map φ is nondegenerate, meaning that the image is not contained in an hyperplane, if and only if the $1, f_1, f_2, \dots, f_r$ are linearly independent.

b) Now assume that $\varphi\colon X \to \mathbb{P}^r$ is a holomorphic map in the form of the previous point. Let $H_i = \{x_i = 0\}$ be the coordinate hyperplanes. Show that

$$\operatorname{div}(f_i) = \varphi^* H_i - \varphi^* H_0$$

Hence, $f_i \in H^0(X, \varphi^* H_0)$. More generally, if $H \subseteq \mathbb{P}^r$ is any hyperplane, show that φ gives $r+1$ linearly independent elements in $H^0(X, \varphi^* H)$.

c) If $H, H', \subseteq \mathbb{P}^r$ are two arbitrary hyperplanes, show that their pullbacks are linearly equivalent: $\varphi^* H \sim \varphi^* H'$. Moreover, if $D \subseteq \mathbb{P}^r$ is an hypersurface of degree d, show that $\varphi^* D \sim d \cdot \varphi^* H$.

Thus, each holomorphic map $\varphi\colon X \to \mathbb{P}^r$ yields the linear system $\{\varphi^* H \mid H \subseteq \mathbb{P}^r \text{ hyperplane}\}$. Equivalently, it yields an $r+1$-dimensional subspace $V \subseteq H^0(X, \varphi^* H)$, where $H \subseteq \mathbb{P}^r$ is any hyperplane.