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Riemann Surfaces and Algebraic Curves
WiSe 2020/21

Exercise Sheet 4

These exercises will be discussed on December 9

Exercise 5.1 (Lifting points on tori) Let $p_1, p_2, \dots, p_d$ and $q_1, q_2, \dots, q_d$ be points on a complex torus $X = \mathbb{C}/L$ for a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$. Show that there are $x_1, x_2, \dots, x_d$ and $y_1, y_2, \dots, y_d$ in $\mathbb{C}$ such that $\overline{x_i} = p_i$, $\overline{y_j} = q_j$ with $\sum x_i = \sum y_j$ if and only if $\sum p_i = \sum q_j$ in the quotient group law on X.

Exercise 5.2 (More on Theta Functions) Let $L = \mathbb{Z} + \mathbb{Z}\tau$ a lattice in $\mathbb{C}$ where $\tau \in \mathbb{C}$ with positive imaginary part.

a) Show that $\theta_\tau(z + a + b\tau) = \exp(-\pi i b^2\tau - 2\pi i b z)\theta_\tau(z)$.

b) Show that $\theta^{(x)}(z + \tau) = -\exp(-2\pi i(z - x)) \cdot \theta^{(x)}(z)$.

c) Let $a, b \in \frac{1}{\ell}\mathbb{Z}$ be rational numbers with denominator that divides $\ell \in \mathbb{N}$. Set

$$\vartheta_{a,b}(z, \tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i(a + n)^2\tau + 2\pi i(n + a)(z + b))$$

Show that $\vartheta_{0,0}(z, \tau) = \theta_\tau(z)$ and, more interestingly, that for all $p, q \in \mathbb{Z}$

$$\vartheta_{a+p,b+q}(z, \tau) = \exp(2\pi i a q) \cdot \vartheta_{a,b}(z, \tau).$$

Exercise 5.3 (Abel's Theorem for a Torus) Let τ be a complex number with positive real part and let L be the lattice $\mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$. Write $X = \mathbb{C}/L$ for the Riemann surface obtained as the quotient. Define the map $A\colon \mathrm{Div}(X) \to X$, $\sum_{p \in X} n_p \cdot p \mapsto \sum n_p \cdot p$, where the second sum is taken with respect to the group structure on X.

a) Show that a divisor D is principal if and only if $\deg(D) = 0$ and $A(D) = 0$.

b) Conclude that two divisors D_1 and D_2 on X are linearly equivalent if and only if $\deg(D_1) = \deg(D_2)$ and $A(D_1) = A(D_2)$.

c) Show that for every divisor D of degree 1 there is a unique point $q \in X$ such that D is linearly equivalent to the divisor q.