Riemann Surfaces and Algebraic Curves
WiSe 2020/21

Exercise Sheet 4

These exercises will be discussed on December 2

Exercise 4.1 (Meromorphic functions II)

a) Consider $f = z^3/(1-z^2)$ as a meromorphic function on $\mathbb{C}_\infty = \mathbb{P}^1$. Find all points p such that $\mathrm{ord}_p(f) \neq 0$. Show that the associated holomorphic map $F\colon \mathbb{P}^1 \to \mathbb{P}^1$ has degree 3. What are its ramification and branch points?

b) Same as part (a) for $f = 4z^2(z-1)^2/(2z-1)^2$. The degree of F here should be 4.

Exercise 4.2 (The double-cover for hyperelliptic curves) Let X be a (affine) hyperelliptic curve defined by $y^2 = h(x)$ for a polynomial $h \in \mathbb{C}[x]$ of even degree with distinct roots. Let $\pi\colon X \to \mathbb{C}$ be the map defined by coordinate projection $\pi(x, y) = x$.

a) Show that the ramification divisor $R_\pi = \sum_{p\in X}[\mathrm{mult}_p(\pi) - 1]p$ of π is the divisor of zeroes $\mathrm{div}_0(y)$ of the meromorphic function y on X. Can you draw a picture giving a geometric intuition for your reasoning?

b) Does (a) also hold if the polynomial h has odd degree?
If h has odd degree $2g + 1$, then $y^2 = h(x)$ defines the curve as usual in one chart, but the other chart is given by $w^2 = k(z)$ for $k(z) = z^{2g+2}h(1/z)$ with isomorphism $\phi(x, y) = (1/x, y/x^{g+1})$ as usual.

c) What is the branch divisor $B_\pi = \sum_{q\in Y} \left[\sum_{p\in\pi^{-1}(q)}(\mathrm{mult}_p(\pi) - 1)\right] q$ of π?

d) Show that $\pi^*(B_\pi) = 2R_\pi$ (as divisors on X).

Exercise 4.3 (Intersection divisors) Let $C \subset \mathbb{P}^2$ be a smooth projective plane curve.

a) If G_1 and G_2 are two homogeneous polynomials ($\neq 0$) on $\mathbb{P}^2$, then $\mathrm{div}(G_1G_2) = \mathrm{div}(G_1) + \mathrm{div}(G_2)$.

b) Compute the intersection divisors of the lines defined by $X = 0$, $Y = 0$, and $Z = 0$ with C, where C is defined by $Y^2Z = X^3 - XZ^2$.

c) Show that the intersection divisor of any two distinct lines in $\mathbb{P}^2$ has degree 1 (on either line). The intersection divisor of a homogeneous polynomial G of degree d with a line has degree d.

d) When is the intersection divisor of a line $aX + bY + cZ = 0$ with a conic C, say $XY = Z^2$, of the form $2p$ for some $p \in C$?