Riemann Surfaces and Algebraic Curves
WiSe 2020/21

Exercise Sheet 3

These exercises will be discussed on November 18

Exercise 3.1 (Plane curves) Let $C \subseteq \mathbb{P}^2$ be a smooth plane curve which is not a line, defined by an homogeneous polynomial $F(X,Y,Z)$ of degree $d \geq 2$. For any line $\ell \subseteq \mathbb{P}^2$ we can restrict the $F(X,Y,Z)$ to ℓ and obtain an homogeneous polynomial on ℓ with d roots (counted with multiplicity), which correspond to the intersection points $C \cap \ell$. We say that ℓ intersects C with multiplicity m at $p \in C \cap \ell$ if the corresponding root has multiplicity m.

a) For any $p \in C$, show that the tangent line T_pC intersects C at p with multiplicity $m \geq 2$. We say that p is a *flex*, if this multiplicity is $m \geq 3$. Show that C has a finite number of flexes and count them appropriately.

b) Let $p_0 \in \mathbb{P}^2 \setminus C$ be any point and let $\pi \colon C \to \mathbb{P}^1$ be the projection from p_0 onto a line in $\mathbb{P}^2$. Show that $\operatorname{mult}_p(\pi)$ is exactly the intersection multiplicity of the line $\ell(p_0,p)$ with C at p. Conclude that if p_0 is a general point, then $\operatorname{mult}_p(\pi) \leq 2$ for any $p \in C$.

c) Consider the homogeneous coordinates X, Y. Show that the quotient $\frac{X}{Y}$ can be considered as a meromorphic function on C and interpret the order of its zeroes and poles geometrically. In general, do the same for the quotient $\frac{L_1(X,Y,Z)}{L_0(X,Y,Z)}$ of two linear homogeneous polynomials.

Exercise 3.2 (Consequences of Riemann-Hurwitz)

a) Let X be a compact Riemann surface with a nonconstant map $f \colon \mathbb{P}^1 \to X$. Show that $X \cong \mathbb{P}^1$.

b) Let $f(z), g(z), h(z) \in \mathbb{C}[z]$ be nonconstant polynomials such that $f^n + g^n = h^n$. Show that $n = 1$ or $n = 2$. *Hint:* consider the Fermat curve $V = \{X^n + Y^n = Z^n\}$ in $\mathbb{P}^2$ and the map $F \colon \mathbb{C} \to V,\ F(z) = [f(z), g(z), h(z)]$.

c) Let X be an hyperelliptic curve with affine model $\{y^2 = (x - x_1)\dots(x - x_{2g+2})\}$ as in Exercise 2.2. What is the genus of X?

Exercise 3.3 (Meromorphic functions)

a) Let X be a compact Riemann surface. Show that all holomorphic functions $f \colon X \to \mathbb{C}$ are constant.

b) Let f be a meromorphic function on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Show that f is rational, that is $f = \frac{G(z)}{H(z)}$, where $G(z)$ and $H(z)$ are two polynomials.

c) Let X be the hyperelliptic curve with affine model $\{y^2 = (x - x_1)\dots(x - x_{2g+2})\}$ as in Exercise 2.2. Observe that y and x can be interpreted as meromorphic functions on X and compute their zeroes and poles.