Riemann Surfaces and Algebraic Curves
WiSe 2020/21

Exercise Sheet 2

These exercises will be discussed on November 11

Exercise 2.1 (Complex tori)

a) Show that the quotient $\mathbb{C}/\mathbb{Z}[i]$ is naturally a Riemann surface. More generally, argue the same for $\mathbb{C}/L$, where $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ for $\mathbb{R}$-linearly independent complex numbers $\omega_1, \omega_2 \in \mathbb{C}$.

b) Show that a quotient $\mathbb{C}/L$ with L as in the previous part is homeomorphic to $S^1 \times S^1$ (as a topological real manifold), in particular it is compact.

c) Let $m \in \mathbb{Z}$. Show that the multiplication map

$$[m] : \mathbb{C}/L \to \mathbb{C}/L, \qquad [z] \mapsto [m \cdot z]$$

is holomorphic. Compute the degree and the ramification points.

Exercise 2.2 (Hyperelliptic curves) Let $f(x) = (x-a_1)(x-a_2)\dots(x-a_{2g+2})$ be a complex polynomial with $2g+2$ distinct roots. We also define the polynomial $g(u) = u^{2g+2}f(1/u) = (1-a_1u)(1-a_2u)\dots(1-a_3u)$.

a) Show that the two affine curves $X_0 = \{(x,y) \,|\, y^2 = f(x)\}$ and $X_1 = \{(u,v) \,|\, v^2 = g(u)\}$ are nonsingular, hence Riemann surfaces.

b) We consider the two open subsets $V_0 = \{(x,y) \in X_0 \,|\, x \neq 0\}$ and $V_1 = \{(u,v) \,|\, u \neq 0\}$. Show that

$$\phi \colon V_0 \longrightarrow V_1, \qquad \phi(x,y) = \left(\frac{1}{x}, \frac{y}{x^{g+1}}\right)$$

is a biholomorphism.

c) Show that we can glue X_0 and X_1 along the open sets U_0, U_1 to obtain a compact Riemann surface $X = X_0 \cup X_1$. Prove that this surface has a natural map $X \to \mathbb{P}^1$ of degree two and compute the ramification points.

Exercise 2.3 (Maps to $\mathbb{P}^1$) Here we consider $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, where $\infty = [0,1]$.

a) We fix the local coordinate z around $0 \in \mathbb{C}$. Let $f \colon U \to \mathbb{P}^1$ be an holomorphic map in a neighborhood of 0 such that $f(0) = \infty$ and $mult_0(f) = e$. Show that in a punctured neighborhood of 0 we can write $f(z) = \frac{g(z)}{z^e}$, where $g(z)$ is holomorphic and $g(0) \neq 0$.

b) Let $F \colon \mathbb{C} \to \mathbb{P}^1$ be an holomorphic map such that $F^{-1}(0) = \{p_1, \dots, p_n\}$ and $F^{-1}(\infty) = \{q_1, \dots, q_m\}$ with $e_i = mult_{p_i}(F)$ and $f_i = mult_{q_i}(F)$. We have an induced map $f \colon \mathbb{C} \setminus \{a_1, \dots, a_n\} \to \mathbb{C}$: show that we can write it as

$$f(z) = \frac{(z-p_1)^{e_1}(z-p_2)^{e_2}\dots(z-p_n)^{e_m}}{(z-q_1)^{f_1}(z-q_2)^{f_2}\dots(z-q_n)^{f_n}} \cdot g(z)$$

where $g \colon \mathbb{C} \to \mathbb{C}$ is holomorphic and never zero.

c) Show that any holomorphic map $F \colon \mathbb{P}^1 \to \mathbb{P}^1$ has the form

$$F \colon [X_0, X_1] \mapsto [F_0(X_0, X_1), F_1(X_0, X_1)]$$

where $F_0, F_1 \in \mathbb{C}[X_0, X_1]$ are two homogeneous polynomials of the same degree and no common roots.